Typical Ranks of Semisimple Graphs

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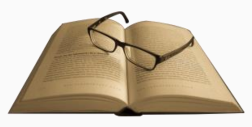
Introductio



M: an $n \times n$ partially-filled symmetric matrix in $\mathcal{S}_n(\mathbb{C})$ or $\mathcal{S}_n(\mathbb{R})$.
→ complete M in a way that the completion $\overline{M}$ has the lowest possible rank (*completion rank* of M).

Ex).

$$M = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}, \quad x = 0 \quad \Rightarrow \quad rank(\overline{M}) = 0$$

M : a partial matrix in $\mathcal{S}_n(\mathbb{C})$
→ complete M in a way that

(1) stable under a perturbation

(2) lowest possible rank (*generic completion rank* of M).

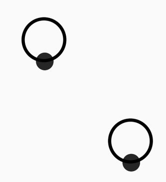
Ex).

$$M = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} \textit{for any } x, \\ rank(\overline{M}) \geq 1 \end{array}$$

$$M' = \begin{bmatrix} 1 \pm \epsilon_1 & x \\ x & 1 \pm \epsilon_2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} \textit{for any } x, \\ rank(\overline{M'}) \geq 1 \end{array} \quad \Rightarrow gcr(M) = 1.$$

For $M \in \mathcal{S}_n(\mathbb{R})$, we call such a rank as a *typical rank*.
Encode M by a semisimple graph G such that $M_{ij} = M_{ji}$ is specified if $(i, j) \in E(G)$.

Ex).



$$\Rightarrow \quad \begin{array}{l} M = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \Rightarrow rank(\overline{M}) \geq 1 \Rightarrow (\textit{minimal typical rank}) = 1 \\ M = \begin{bmatrix} 1 & x \\ x & -1 \end{bmatrix} \Rightarrow rank(\overline{M}) \geq 2 \Rightarrow (\textit{maximal typical rank}) = 2 \end{array}$$

Prop (Prop 6.1. Bernstein-Blekherman-Sinn 2018). *Let $G = (V, E)$ be a semisimple graph with $|G| = n$. Then,*

(1) $gcr(G)$ exists (completion rank of almost all G-partial matrix)

(2) $gcr(G) \geq k$ if $nk - \binom{k}{2} \geq |E|$

(3) the smallest typical rank of $G = gcr(G)$

(4) If $r_1 < r_2$ are typical ranks of G, then so is r such that $r_1 \leq r \leq r_2$

(5) (maximal typical rank of G)$\leq 2 \cdot gcr(G)$

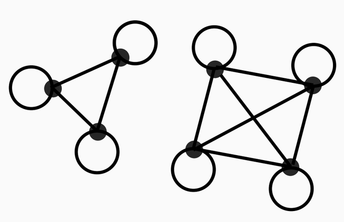
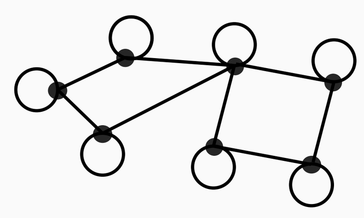
Remark. *By (3) ~ (5), in order to know all typical ranks, it is enough to find the generic completion rank and maximal typical rank.*

Playground!

1 Full-rank Typicality

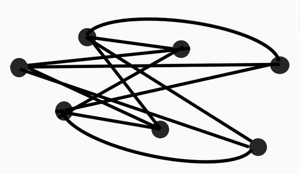
Def. A semisimple graph $G = (V, E)$ is called *full-rank typical* if $|V|$ is a typical rank of G.

Thm (Characterization of full-rank typical graphs). *A semisimple graph G is full-rank typical if and only if the complement G^C of G is bipartite.*

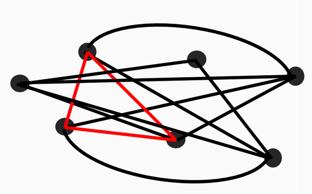
$G =$  $H =$ 

Q. Are G and H full-rank typical?

A.

$G^C =$ 

⇒ full-rank typical

$H^C =$ 

⇒ not full-rank typical

2 Case Study $K_n^\circ \cup K_m^\circ$

Prop. *The typical ranks of $K_n^\circ \cup K_m^\circ$ are $\max\{n, m\}, \ldots, n + m$.*

Def. Given real symmetric matrices A and B of full-rank (possibly different size), we define the *eigenvalue sign disagreement* of A and B as :

$$\mathrm{esd}(A, B) := \begin{cases} 0 & \text{if } (p_A - p_B)(n_A - n_B) \geq 0 \\ \min\{|p_A - p_B|, |n_A - n_B|\} & \text{o.w} \end{cases}$$

where $p_\cdot$ ($n_\cdot$) is the number of positive (negative) eigenvalues.

Thm. *Let $M = \begin{bmatrix} A & X \\ X^\top & B \end{bmatrix}$ be a $K_n^\circ \cup K_m^\circ$-partial matrix where A is an $n \times n$-matrix, B is an $m \times m$-matrix which are full-rank and X is an $n \times m$-matrix with unspecified entries. Then, M is minimally completable to rank $\max\{n, m\} + \mathrm{esd}(A, B)$.*

G-partial matrix in 1

$$M_G = \left[\begin{array}{cccc|ccc} 1 & & & O & & & \\ & 1 & & & & X & \\ & & -1 & & & & \\ O & & & -1 & & & \\ \hline & & & & 1 & & O \\ & X^\top & & & & 1 & \\ & & & & O & & 1 \end{array}\right]$$

Q. What is the minimal typical rank of M_G?

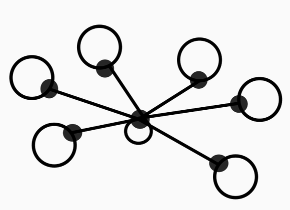
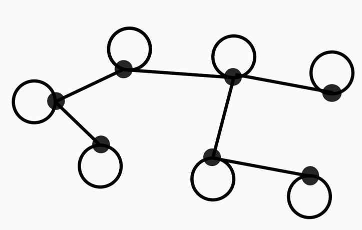
A.

$$\left[\begin{array}{cccc|ccc} 1 & & & O & & & \\ & 1 & & & & X & \\ & & -1 & & & & \\ O & & & -1 & & & \\ \hline & & & & 1 & & O \\ & X^\top & & & & 1 & \\ & & & & O & & 1 \end{array}\right]$$

$\mathrm{esd}(A, B) = 1$

⇒ (minimal typical rank) = 5

3 Small Typical Ranks

$G =$  $H =$ 

Q. What are the maximal typical rank of G (star tree) and H (tree)?

Prop. *Let G be an all-looped semisimple graph. Then the maximal typical rank of G is 3 if and only if G is either*

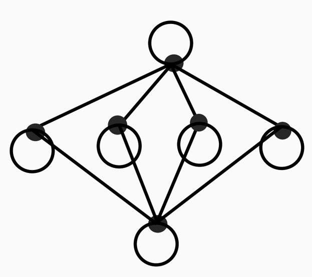
(1) a triangle or

(2) a disjoint union of a star tree and a (possibly empty) set of isolated vertices.

Thm (Typical ranks of all-looped trees). *All-looped trees have typical rank at most 4.*

A. (maximal typical rank of G)= 3
(maximal typical rank of H)= 4

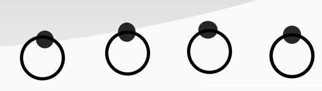
4 Bounds on Typical Ranks

$G =$ 

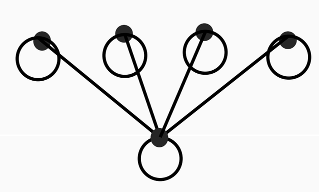
Q. Find the upper bound of the typical rank of G.

Prop. *Let G be an all-looped semisimple graph and r be the size of the maximum independent set of G. If G has n vertices, then the maximal typical rank of G is at most $2 + (n - r)$.*

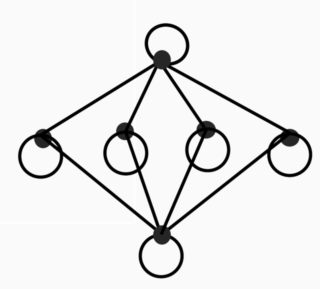
A.



(typical rank) ≤ 2



(typical rank) ≤ 3



(typical rank) ≤ 4