Example). $\lim_{x\to0}\frac{\sin 2x}{x} = \lim_{x\to0} 2\,\frac{\sin 2x}{2x} = 2\lim_{x\to0} 1 = 2.$

$$\lim_{x\to0}\frac{\sin 4x}{5x} = \lim_{x\to0}\frac{\sin 4x}{4x}\cdot\frac{4x}{5x} = \frac{4}{5}\lim_{x\to0}\frac{\sin 4x}{4x} = \frac{4}{5}.$$

$$\lim_{x\to0}\frac{\sin ax}{bx} = \lim_{x\to0}\frac{\sin ax}{ax}\cdot\frac{ax}{bx} = \frac{a}{b}\lim_{x\to0}\frac{\sin ax}{ax} = \frac{a}{b}.$$

$$\lim_{x\to0}\frac{\cos \cancel{b}x-1}{\cancel{b}x}$$ $\ast\ \cos h = 1-2\sin^2\left(\frac{h}{2}\right).$

$$= \lim_{x\to0}\frac{-2\sin^2\left(\frac{\cancel{b}x}{2}\right)}{x} = \lim_{x\to0}\frac{-2\sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}{\frac{x}{2}\cdot 2}.$$

$$= \lim_{x\to0}\frac{-\sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = -\lim_{x\to0}\sin\left(\frac{x}{2}\right)\lim_{x\to0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = 0.$$

$$\lim_{x\to0}\frac{\tan x\sec 2x}{3x} = \lim_{x\to0}\frac{\frac{\sin x}{\cos x}\cdot\frac{1}{\cos 2x}}{3x} = \lim_{x\to0}\frac{\sin x}{3x\cos x\cos 2x}$$

$$= \frac{1}{3}.$$

4.2.5. continuity.

* continuity at a point.

$f(x)$ is continuous at $x=c$ if.

$$\lim_{x\to c} f(x) = f(c).$$

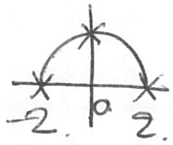
$f(x)$ is right-continuous at c if.

$$\lim_{x\to c^+} f(x) = f(c).$$

$f(x)$ is left-continuous at c if.

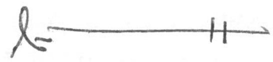
$$\lim_{x\to c^-} f(x) = f(c).$$

ex). $f(x) = \sqrt{4-x^2}$.



$f(x)$ is continuous at $x=0$.

$f(x)$ is r-continuous at $x=-2$.



$x=2$.

* continuous on a closed interval.

  $f(x)$ is continuous on $[a,b]$.

  left-cont at b.

  r-cont at a.

  cont on (a,b).

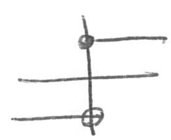
* Continuous function.

  : continuous on its domain.

  ex) $y=x^2$. ~~on $(-\infty,\infty)$~~.

  $y=\frac{1}{x}$. ?

  $y=\begin{cases} 1, & x\geq 0 \\ -1, & x<0. \end{cases}$



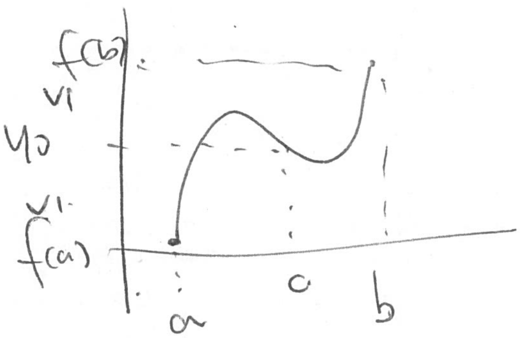
  $y=\begin{cases} 1, & x\geq 0 \\ -1, & x<0. \end{cases}$ on $(-\infty,-1)$.

<u>Remark</u>. polynomial, rational, inverse, composite of conts are continuous.

* The Intermediate Value Theorem. (IVT).

f: continuous on $[a,b]$, $f(a) \le y_0 \le f(b)$.

$\Rightarrow$ $y_0 = f(c)$ for some c in $[a,b]$.



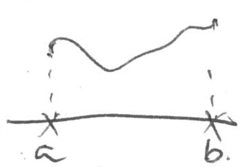
Ex). f: a positive continuous function on $[a,b]$

f: never zero on $[a,b]$.

T/F?

f does not change sign on $[a,b]$.

T



If not,

then there is c in (a,b)

that $f(c) < 0$.

then, since $f(c) \le 0 \le f(a)$,

there is $d \in (a,c)$ s.t.

$f(d) = 0$.

Example). Show that there is a root of $x^3 - x - 1 = 0$

on $[1,2]$

$f(x) = x^3 - x - 1$. $f(1) = -1 < 0$.

$f(2) = 8 - 2 - 1 = 5 > 0$.

there is c in $(1,2)$, $f(c) = 0$.

* Continuous Extension..

make discontinuous function continuous.

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ c, & x = 0. \end{cases}$$

Find the value of c which makes F(x) continuous.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1..$$

$$\Rightarrow \lim_{x \to 0} F(x) = 1. \qquad F(0) = 1 = c.$$

Example) $f(x) = \dfrac{x^2+x-6}{x^2-4}$, $x \neq 2$.

Find a value f(2) which makes f(x) cont.

$$f(x) = \frac{(x+3)(x-2)}{(x-2)(x+2)} = \frac{x+3}{x+2} \quad \text{if } x \neq 2.$$

$$\lim_{x \to 2} f(x) = f(2) \quad \text{if } f \text{ is cont.}$$

$$\Rightarrow f(2) = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

§2.6. Limits Involving Infinity.

* Limits as $x \to \pm\infty$.

$f(x)$ has the limit L as $x \to \infty$.

$$\lim_{x\to\infty} f(x) = L.$$

$f(x)$ has the limit L as $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = L.$$

Remark. Limit Laws work on $\lim_{x\to\infty} f(x)$ also.

ex) $\lim_{x\to\infty}\left(\frac{1}{x}+5\right) = \lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}5 = 5.$

Recall. Rational functions.

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} = \frac{5}{3}.$$

* Horizontal Asymptotes.

A line $y=b$ is a horizontal asymptotes of $f(x)$ if $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

ex) $y = \frac{1}{x}+1$

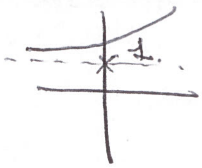
$\lim_{x\to\infty}\left(\frac{1}{x}+1\right) = 1.$

$\Rightarrow$ $y=1$. horizontal asymptotes.

Finding Horizontal Asymptotes.

~~Dividing~~ ex). $f(x) = \frac{x^3-2}{|x|^3+1} = \begin{cases} \frac{x^3-2}{x^3+1} & \text{if } x \geq 0 \\ \frac{x^3-2}{-x^3+1} & \text{if } x < 0. \end{cases}$

ex) $\lim_{x\to-\infty} e^{x}+1 = 1.$



ex) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0} \sin(t) = 0.$

$t=\frac{1}{x}.$

ex). $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0} \frac{1}{t}\sin(t) = 1.$

$t=\frac{1}{x}.$

ex) $\lim_{x\to\infty} \left(x-\sqrt{x^2+16}\right) = \lim_{x\to\infty} \left(x-\sqrt{x^2+16}\right)\left(\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}\right)$

$= \lim_{x\to\infty} \frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} = \lim_{x\to\infty} \frac{-16}{x+\sqrt{x^2+16}} = 0.$

* Oblique Asymptotes.

$y=mx+b$ $(m\neq 0)$ is an oblique asymptote if $f(x)$ approaches to $y=mx+b$ as $x\to\pm\infty$.

∴ (degree of numerator) = (degree of denominator) +1.

ex). $\frac{x^2+3}{x-2}$. Find the oblique asymptote.

∴ long division

$$\begin{array}{r|l} & x+2 \\ \hline x-2 & x^2+3 \\ & x^2-2x \\ \hline & 2x+3 \\ & 2x-4 \\ \hline & 7 \end{array}$$

$\frac{x^2+3}{x-2} = \boxed{x+2} + \frac{7}{x-2}.$ oblique asymptote.

$\lim_{x\to\infty}\left[(x+2)+\frac{7}{x-2}\right] \longrightarrow (x+2)$

as $x\to\infty$.

$\frac{-3x^2+2}{x-1}.$

* vertical asymptote.

A line $x=a$ is a vertical asymptote of $f(x)$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

~~Exa~~ ex) $\lim_{x\to 0^+} \frac{1}{x}+2 = +\infty$, $\lim_{x\to 0^-} \frac{1}{x}+2 = -\infty$.

vertical asymptote $x=0$.

ex) ~~lim~~ $\frac{1}{x-2}$. $\lim_{x\to 2^+} \frac{1}{x-2} = +\infty$.

vertical asymptote $x=2$.

zeros on denominator

ex) $f(x) = -\frac{8}{x^2-4}$.

$x^2-4 = (x-2)(x+2)$

$x = \pm 2$.

$\lim_{x\to 2^+} -\frac{8}{x^2-4} = -\infty$. $\lim_{x\to 2^-} -\frac{8}{x^2-4} = -\infty$.

vertical asymptote $\underline{x = \pm 2}$

ex). $\lim_{x\to 0^+} \ln x = -\infty$.

$x=0$.

ex). ~~secx~~

Chapter 3. Derivatives.

§3.1. Tangents and the Derivative at a point.

Recall. $f(x)$: function.

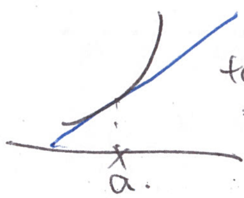
$[a,b]$: an interval.

- Average rate of change.

secant line.

$\Rightarrow$ (slope of secant) $= \dfrac{f(b)-f(a)}{b-a}$

- Instantaneous rate of change.



tangent line.

$\Rightarrow$ (slope of tangent) $= \dfrac{f(a+h)-f(a)}{h}$

Small h

*. Finding tangent.

$$(\text{Slope of the curve}) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = m.$$

: slope at $(x_0, f(x_0))$.

(Tangent line) = a line through $(x_0, f(x_0))$ with the slope m.

Example) $f(t) = 3 - t^2 + 2t$.

1). Find the slope at $x=a$.

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[3-(a+h)^2+2(a+h)]-[3-a^2+2a]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3}-\cancel{a^2}-2ah-h^2+\cancel{2a}+2h-\cancel{3}+\cancel{a^2}-\cancel{2a}}{h}$$

$$= \lim_{h\to 0} \frac{-h^2-2ah+2h}{h} = \lim_{h\to 0} -h-2a+2 = -2a+2.$$

2). Find $x=a$ with the slope 4.

$-2a+2=4.$ $a=-1$ ⟶ a function ; derivative.

3) Find the tangent line at $a=2$.

$f(2)=3-4+4=3 \Rightarrow (2,3)$ (point).

$-2a+2=-2$ (slope)

$y=-2x+b$

$\Rightarrow 3=-4+b. \Rightarrow b=7.$ $\underline{y=-2x+7}$.

* Derivative at a point.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

: The derivative of f at x. if limit exists.

ex) $y=12\sqrt{x}$. tangent at (1,12).

$$\lim_{h\to 0}\frac{12\sqrt{x+h}-12}{h}$$

ex?. $V=\frac{4}{3}\pi r^3$ (volume of a ball with radius r)

Find the rate of change of the volume at $r=4$.

§ 3.2. Derivative as a function.

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is a function of x.

If limit exists, we say $f(x)$ is differentiable.

$$f'(x)=\frac{d}{dx}f(x)=\frac{df}{dx}$$

Example). Find the derivative of $y=\sqrt{x}$.

$$y'=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

ex) $y=\frac{9}{\sqrt{x+1}}$

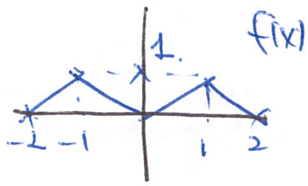
$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\frac{9}{x+h+1}-\frac{9}{x+1}}{h}=\lim_{h\to 0}\frac{\frac{\cancel{9x}+\cancel{9}-\cancel{9x}-9h-\cancel{9}}{(x+h+1)(x+1)}}{h}=\frac{-9h}{h(x+h+1)(x+1)}$$

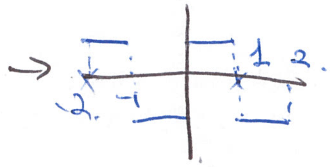
=

* Graphing the Derivative.

$f'(x) = $ (Slope of $f(x)$ at x)

ex)



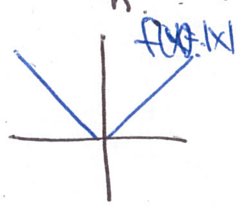


* One-Sided Derivative.

$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$: Right-hand derivative at a

$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$: Left-hand derivative at a.

ex)



$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1.$

$\neq$

$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ not exists. $\Rightarrow$ not differentiable at a.

* Differentiability.

$f(x)$: differentiable on an open interval (a, b).

: differentiable at all points in (a, b).

$f(x)$: differentible on a closed interval $[a, b]$.

: differentiable on (a, b).

Left-hand derivative at $x = b$ exists.

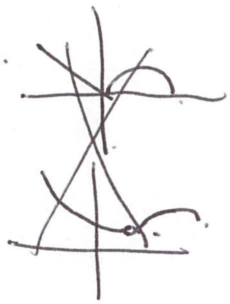
right-hand derivative at $x = a$ exists.

* Differentiability and Continuity.

(Differentiable) $\Rightarrow$ (continuous).

Converse?

example?  $y=|x|$. Consider the example.

ex). 

ex). Find an function continuous on real line but not differentiable at $x=3$.

$y=|x-3|$.

§3.3. Differentiation Rules.

* Derivative Rules.

Constant. $\frac{d}{dx}c=0$. 

Sum rule $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$.

constant multiple $\frac{d}{dx}cf(x) = cf'(x)$.

power rule $\frac{d}{dx}(x^n) = nx^{n-1}$.

Product rule. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x)+f(x)g'(x)$.

quotient rule. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$.

e^x $\frac{d}{dx}e^x = e^x$.

Example) $y(x)=\dfrac{x^2+12}{2x-11} \Rightarrow y'(x)?$

$$y'(x)=\frac{(2x)(2x-11)-2(x^2+12)}{(2x-11)^2}.$$

Example) $y=x^{\frac{9}{4}}+e^{-2x}=$

$$y=x^{\frac{9}{4}}+\frac{1}{e^{2x}} \Rightarrow y'=\frac{9}{4}x^{\frac{5}{4}}+\frac{0\cdot e^{2x}-[e^{2x}]'1}{[e^{2x}]^2}$$

$$=\frac{9}{4}x^{\frac{5}{4}}+\frac{-[(e^x)'e^x+e^x\cdot(e^x)']1}{e^{4x}}$$

$$=\frac{9}{4}x^{\frac{5}{4}}+\frac{-2e^{2x}}{e^{4x}}=\frac{9}{4}x^{\frac{5}{4}}-\frac{2}{e^{2x}}.$$

* Higher-order Derivative.

$$f''(x)=\frac{d}{dx}\left(\frac{df}{dx}\right)=\frac{d^2f}{dx^2}.$$

$$f^{(n)}(x)=\frac{d}{dx}\left(\frac{d^{n-1}f}{dx^{n-1}}\right)=\frac{d^nf}{dx^n}.$$

ex) find all Higher-order derivatives of $y=x^3-3x^2+2$.

$y'=3x^2-6x$

$y''=6x-6$

$y'''=6$

$y^{(4)}=0.$

$y^{(5)}=0$

$\vdots$

§3.4. The Derivative as a Rate of change.

Recall. $f(x)$: a function.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$: the instan. rat. chg of f at x_0

if limit exists.

* Motion along a line.

~~velocity~~. $f(t)$: a position at time t. $(s(t))$.

(Velocity) $= v(t) = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$.

(Speed) $= |v(t)|$ no direction.

(acceleration) $= a(t) = \frac{dv}{dt} = \frac{d^2f}{dt^2}$.

Example). $f(t) = t^2 - 4t + 3$. on $0 \le t \le 7$.

1). Position, at $t=7$, average velocity.

2). speed and acceleration at $t=7$.

1) $f(7) = 49 - 28 + 3 = 24$.

(aver. velo) $= \frac{f(7)-f(0)}{7} = \frac{24-3}{7} = 3$.

2). $f'(t) = 2t - 4$

$\Rightarrow$ (speed) $= |f'(7)| = |10| = 10$.

(acceleration) $= f''(t) = \underline{2}$.

§ 3.6. Derivatives of Trigonometric Functions.

$\frac{d}{dx}\sin x = \cos x$ $\qquad$ $\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}\cos x = -\sin x.$ $\qquad$ $\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx}\cot x = -\csc^2 x.$

Example) $y = 7x^2 \sin x + 14x\cos x - 14\sin x$

$\frac{dy}{dx} = 14x\sin x + 7x^2\cos x + 14\cos x - 14x\sin x - 14\cos x.$

$y = \frac{\cot x}{1+\cot x}$

$$\frac{dy}{dx} = \frac{-\csc^2 x\,(1+\cot x) - [-\csc^2 x \cdot \cot x]}{(1+\cot x)^2}$$

$$= \frac{-\csc^2 x}{(1+\cot x)^2}.$$

$y = 2\csc x$

y'' ?

$y' = -2\csc x \cot x$

$y'' = -2(\csc x \cot^2 x + \csc x(-\csc^2 x))$

$= -2\csc x(\cot^2 x - \csc^2 x)$

§ 3.6. The Chain Rule.

Q: How to differentiate. $\underline{F(x) = \sin(x^2-4)}$?

composite function.

ex). $\cancel{y = \frac{3}{2}x}$, $y = 3u = f(u)$, $u = \frac{1}{2}x = g(x)$, $(f \circ g)(x) = \frac{3x}{2}$.

$y \circ u$ $\cancel{g \circ f}$ $(f \circ g)(x)$

$f'(u) = \frac{df}{du} = 3$. $g'(x) = \frac{dg}{dx} = \frac{1}{2}$. $\cancel{(g \circ f)}(x) = \frac{3}{2} = \frac{df}{du} \cdot \frac{dg}{dx}$.

$(f \circ g)'(x) =$

* Chain Rule.

$f(u)$: differentiable at $u = g(x)$.

$g(x)$: differentiable at x.

$(f \circ g)(x) = f(g(x))$ is differentiable at x.

and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

ex). $x(t) = \cos(t^2+1)$. Find $x'(t)$.

$\cancel{\cos u}$ $f(u) = \cos u$.

$u = g(t) = t^2+1$.

$x(t) = (f \circ g)(t) \Rightarrow x'(t) = f'(g(t)) \cdot g'(t)$.

$f'(u) = -\sin u, \Rightarrow \quad = -\sin(t^2+1) \cdot 2t$.

$g'(t) = 2t$.

ex). $\left(\sin(x^2+e^x)\right)'$?

$f(x) = \sin x$

$g(x) = x^2 + e^x$.

$[\sin(x^2+e^x)]' = [(f \circ g)(x)]' = f'(g(x))\, g'(x)$

$= \cos(x^2+e^x) \cdot (2x+e^x)$

ex). $g(t) = \tan(5 - \sin 2t)$.

$\cancel{g(t)} =$ $f(t) = \tan t$
$h(t) = 5 - \sin 2t$.

$g'(t) = (f \circ h)'(t) = f'(h(t)) \cdot h'(t) = \sec^2(5 - \sin 2t) \cdot [5 - \sin 2t]'$

$[5 - \sin 2t]' = [-\sin 2t]'$.

$f(t) = \sin t$ $\quad [-\sin 2t]' = (f \circ h)'(t) = f'(h(t))h'(t)$.
$h(t) = 2t$.
$= -\cos(2t) \cdot 2$.

$\Rightarrow g'(t) = -2\cos(2t)\sec^2(5 - \sin 2t)$

* Chain rule with powers.

Recall, $\frac{d}{dx} x^n = nx^{n-1}$. (power rule).

ex) (a). $\frac{d}{dx}(5x^3 - x^4)^7 \Rightarrow$ $f(t) = t^7$. $\Rightarrow$ $f'(t) = 7t^6$.
$g(x) = 5x^3 - x^4$. $\quad g'(x) = 15x^2 - 4x^3$.

$\Rightarrow \frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \cdot (15x^2 - 4x^3)$.

(b). $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$.

$\Rightarrow$ $f(t) = t^{-1}$. $\Rightarrow$ $f'(t) = -t^{-2}$.
$g(x) = 3x - 2$. $\quad g'(x) = 3$.

$\Rightarrow \frac{d}{dx}(3x-2)^{-1} = -(3x-2)^{-2} \cdot 3 = \frac{-3}{(3x-2)^2}$.

(c). $\frac{d}{dx} e^{\sqrt{3x+1}} \Rightarrow$ $f(x) = e^x$ $\quad \Rightarrow$ $f'(x) = e^x$
$g(x) = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$. $\quad g'(x) = \frac{1}{2}(3x+1)^{-\frac{1}{2}} \cdot 3$.
$= \frac{3}{2\sqrt{3x+1}}$.

$\Rightarrow \frac{d}{dx} e^{\sqrt{3x+1}} = e^{\sqrt{3x+1}} \cdot \frac{3}{2\sqrt{3x+1}}$.

§3.7 Implicit Differentiation.

Motivation $y=x^2-1$. $y^2=x-1$.

explicit form $y=f(x)$.

Implicit form. functions not in $y=f(x)$ form. (usually $F(x,y)=0$ form).

Differentiation?

$y=f'(x)$???

$\longrightarrow$ consider y as a function of x!

$y^2=x-1 \Rightarrow \{y(x)\}^2=x-1$.

$\longrightarrow$ $2y(x)\cdot \frac{dy}{dx}(x)=1$.

$\frac{dy}{dx}=\frac{1}{2y}$.

ex) Find the slope of ~~the circle~~ $x^2+y^2=25$ at $(3,-4)$.

(slope) $=\frac{dy}{dx}$.

$x^2+y^2=25 \Rightarrow 2x+2y\frac{dy}{dx}=0$.

$\Rightarrow 2y\frac{dy}{dx}=-2x \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$.

$\Rightarrow \frac{dy}{dx}(3,-4)=-\frac{3}{(-4)}=\frac{3}{4}$.

ex) $y^2=x^2+\sin(xy)$. Find $\frac{dy}{dx}$?

$2y\cdot\frac{dy}{dx}=2x+\cos(xy)(xy)'$

$2y\frac{dy}{dx}=2x+\cos(xy)\cdot\left(y+x\frac{dy}{dx}\right)=2x+y\cos(xy)+x\frac{dy}{dx}\cos(xy)$.

$\Rightarrow 2y\frac{dy}{dx}-x\frac{dy}{dx}\cos(xy)=2x+y\cos(xy)$

$\Rightarrow \frac{dy}{dx}(2y-x\cos(xy))=2x+y\cos(xy) \Rightarrow \frac{dy}{dx}=\frac{2x+y\cos(xy)}{2y-x\cos(xy)}$.

ex). $\frac{d^2y}{dx^2}$? if $2x^3-3y^2=8$.

$$6x^2-6y\cdot\frac{dy}{dx}=0. \Rightarrow \frac{dy}{dx}=\frac{6x^2}{6y}=\frac{x^2}{y}.$$

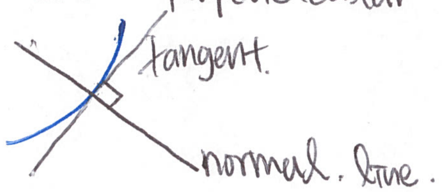
$$\Rightarrow 12x-6\left(\frac{dy}{dx}\cdot\frac{dy}{dx}+y\frac{d^2y}{dx^2}\right)=0. \Rightarrow 12x-6\left(\frac{dy}{dx}\right)^2-6y\frac{d^2y}{dx^2}=0.$$

$$\Rightarrow 12x-6\frac{x^4}{y^2}-6y\frac{d^2y}{dx^2}=0 \Rightarrow -6y\frac{d^2y}{dx^2}=\frac{6x^4}{y^2}-12x$$

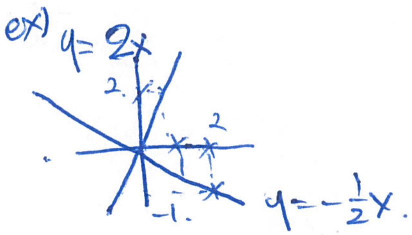
$$\Rightarrow \frac{d^2y}{dx^2}=-\frac{x^4}{y^3}+\frac{2x}{y}.$$

* normal line.

a line perpendicular to the tangent line.



~~ex~~. ~~Find~~. ~~ta~~ (slope of normal line) = $-\frac{1}{(\text{slope of tangent line})}$



ex) Find the normal line of $x^3+y^3-9xy=0$ at $(2,4)$.

(slope of the tangent) = $\frac{dy}{dx}$.

$$3x^2+3y^2\frac{dy}{dx}-9y-9x\frac{dy}{dx}=0.$$

$$\Rightarrow \frac{dy}{dx}(3y^2-9x)=9y-3x^2.$$

$$\Rightarrow \frac{dy}{dx}=\frac{9y-3x^2}{3y^2-9x}=\frac{3y-x^2}{y^2-3x}. \Rightarrow \frac{dy}{dx}(2,4)=\frac{12-4}{16-6}=\frac{8}{10}=\frac{4}{5}.$$

$\Rightarrow$ (slope of normal) $=-\frac{5}{4}$. $\Rightarrow$ (normal) = $y-4=-\frac{5}{4}(x-2)$.