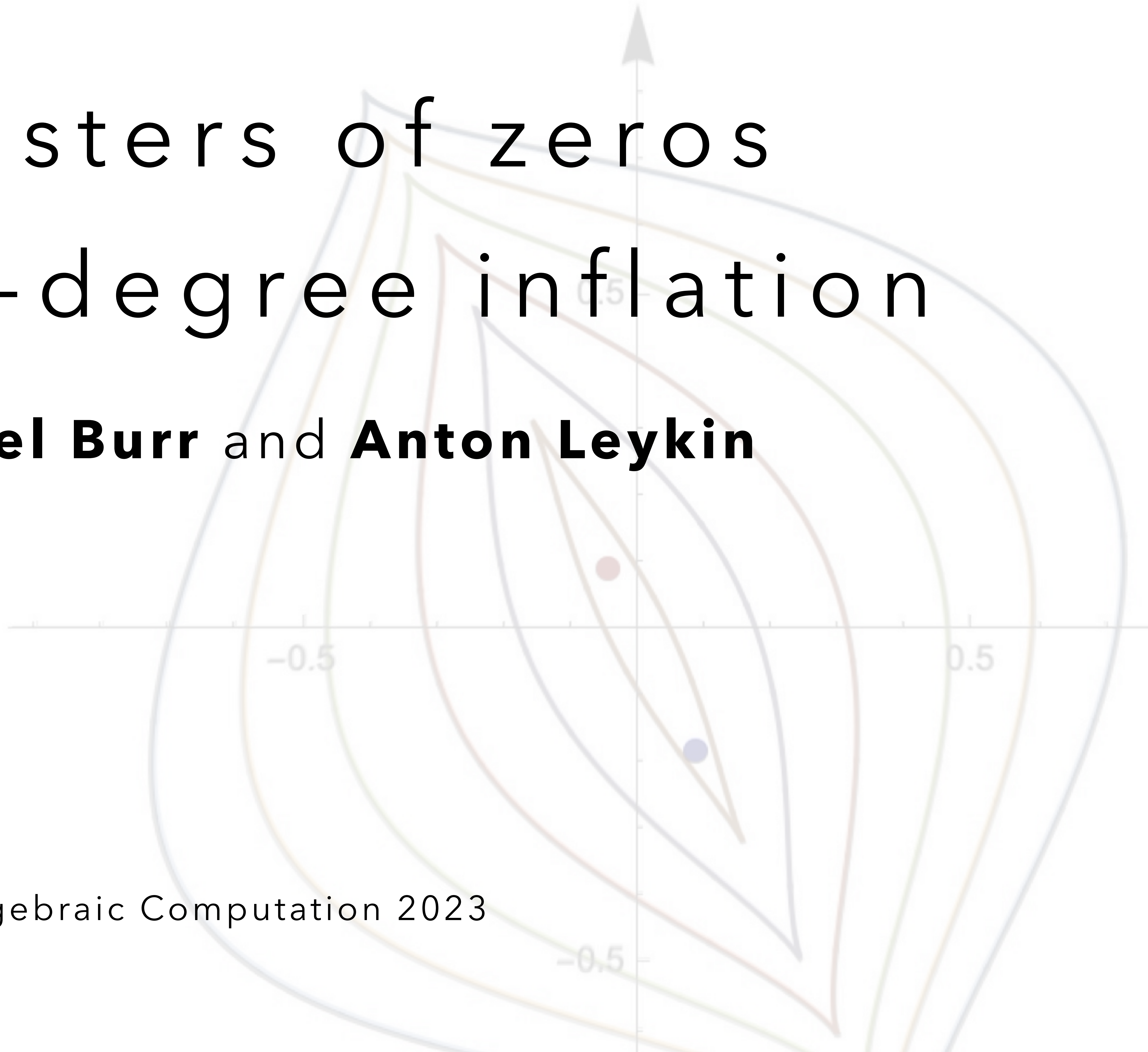
Isolating clusters of zeros using arbitrary-degree inflation

Joint work with **Michael Burr** and **Anton Leykin**



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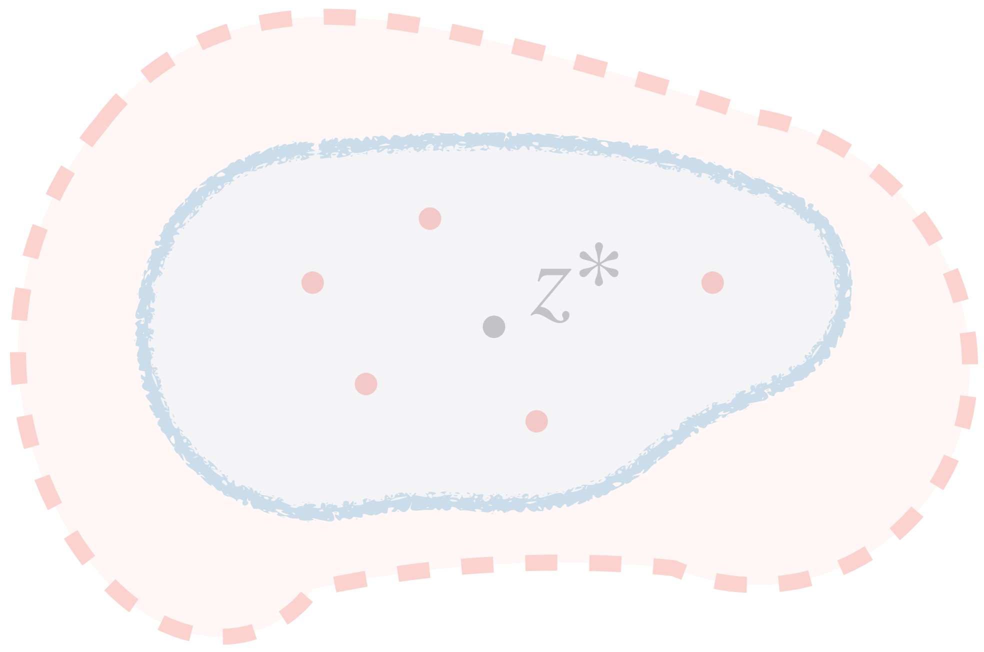
Zero cluster isolation problem

$\mathscr{F}$: a system of analytic functions

$z^* \in \mathbb{C}^n$: a point approximating zeros of $\mathscr{F}$

The **zero cluster isolation problem** is to compute two regions R_- and R_+, and a positive integer c such that



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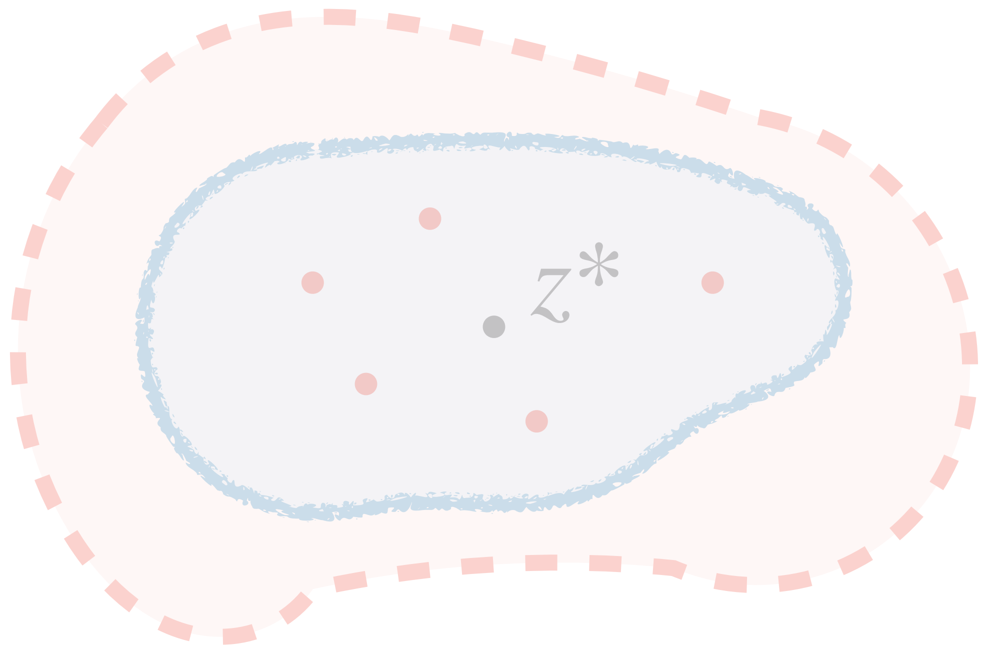

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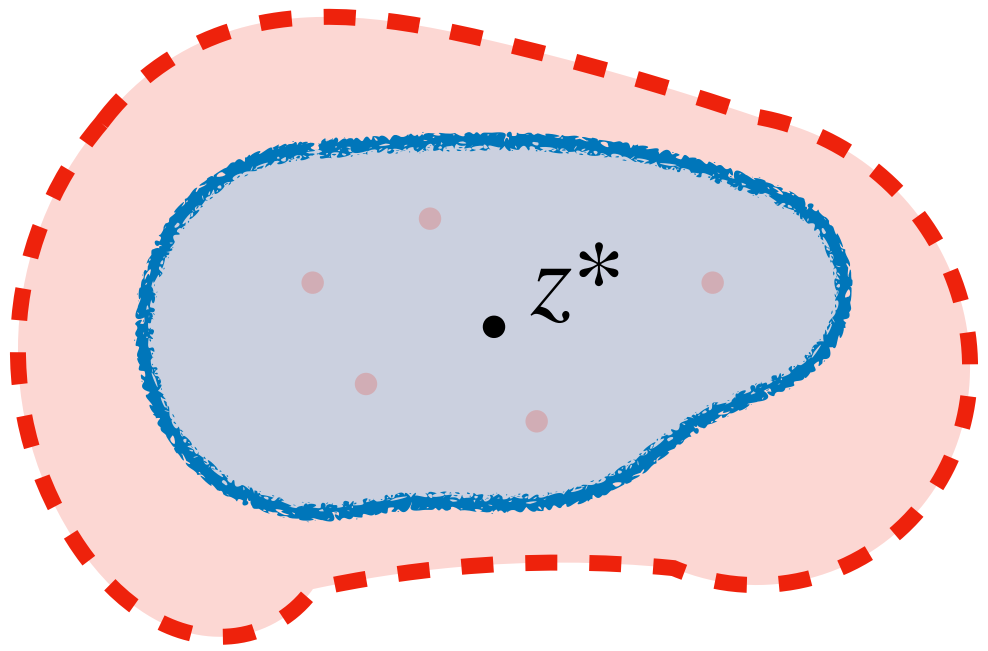

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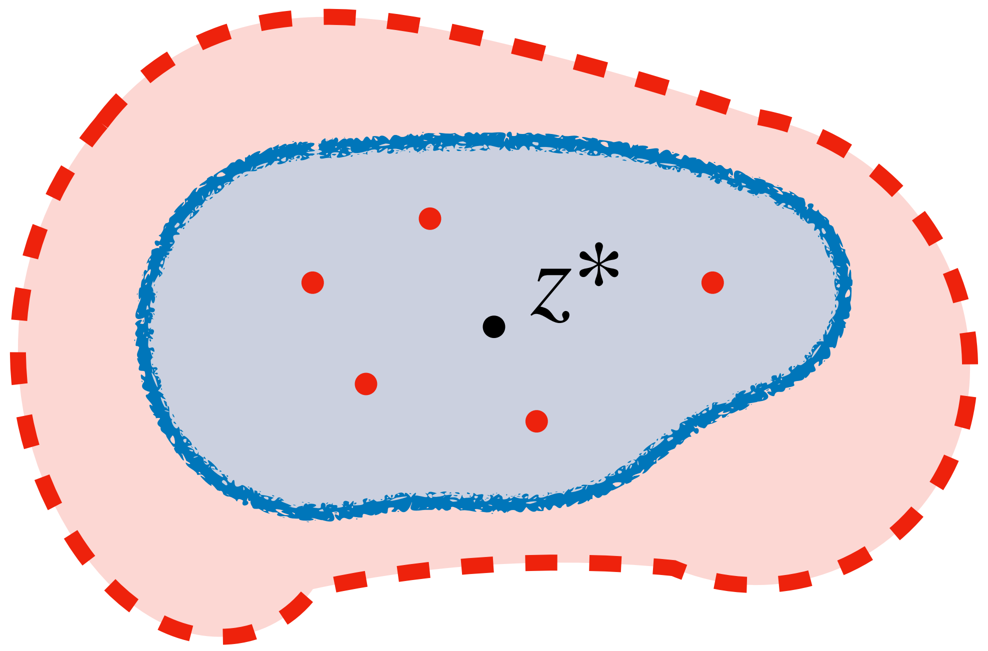

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Zero cluster isolation problem

- Solving a system with a multiple zero using a numerical solver generates a cluster of zeros
- R_- tightly encloses the cluster
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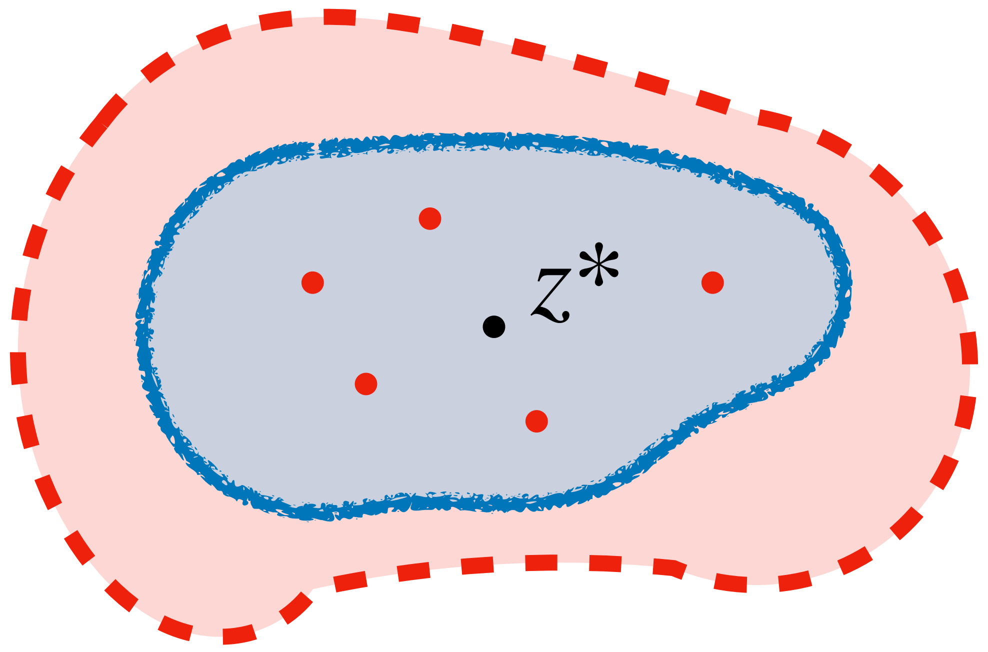

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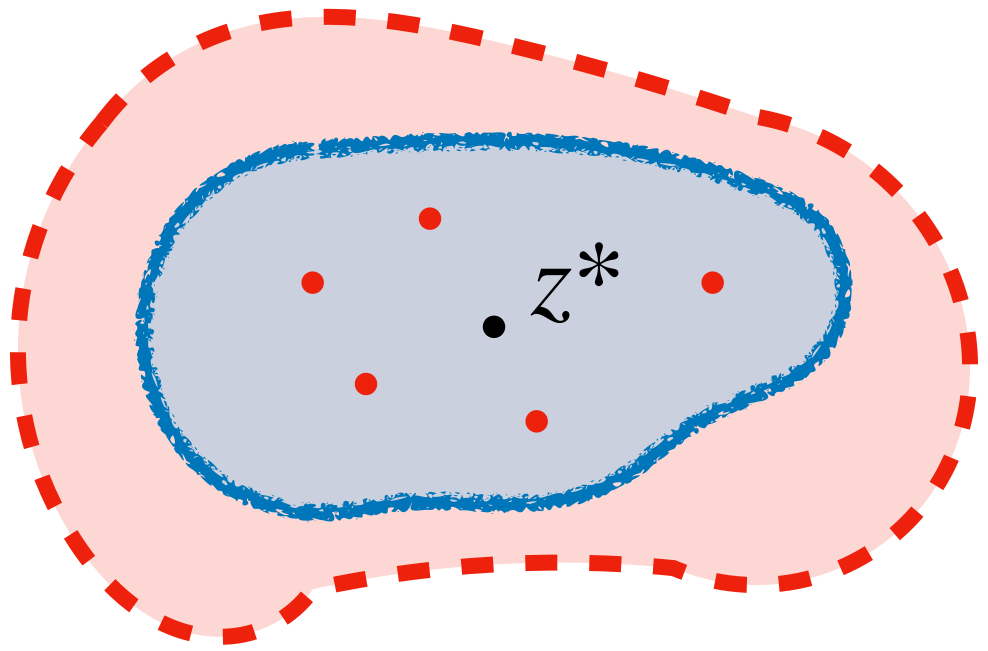

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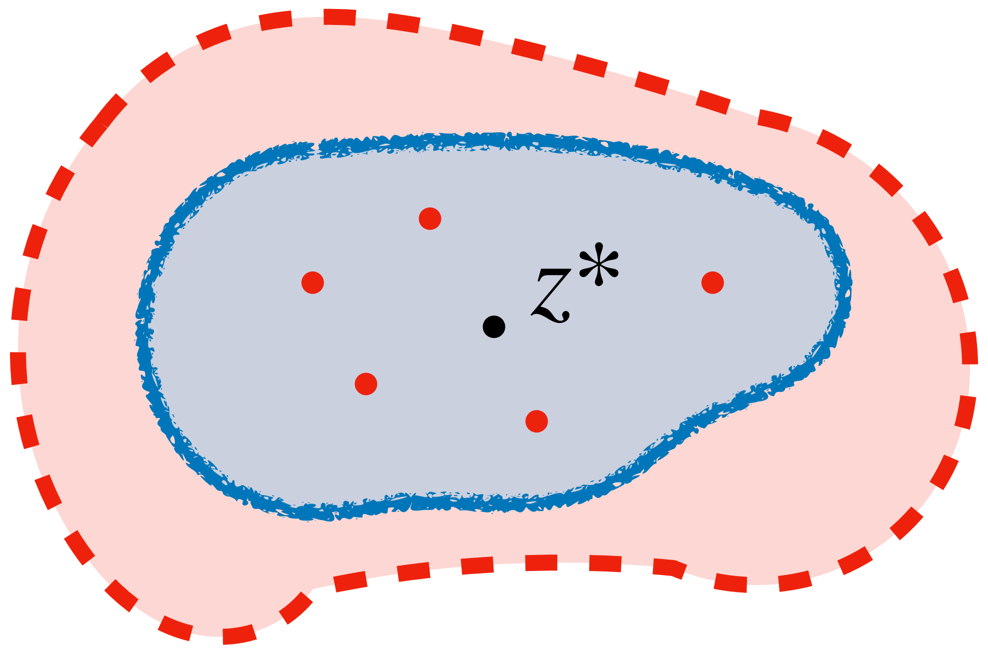

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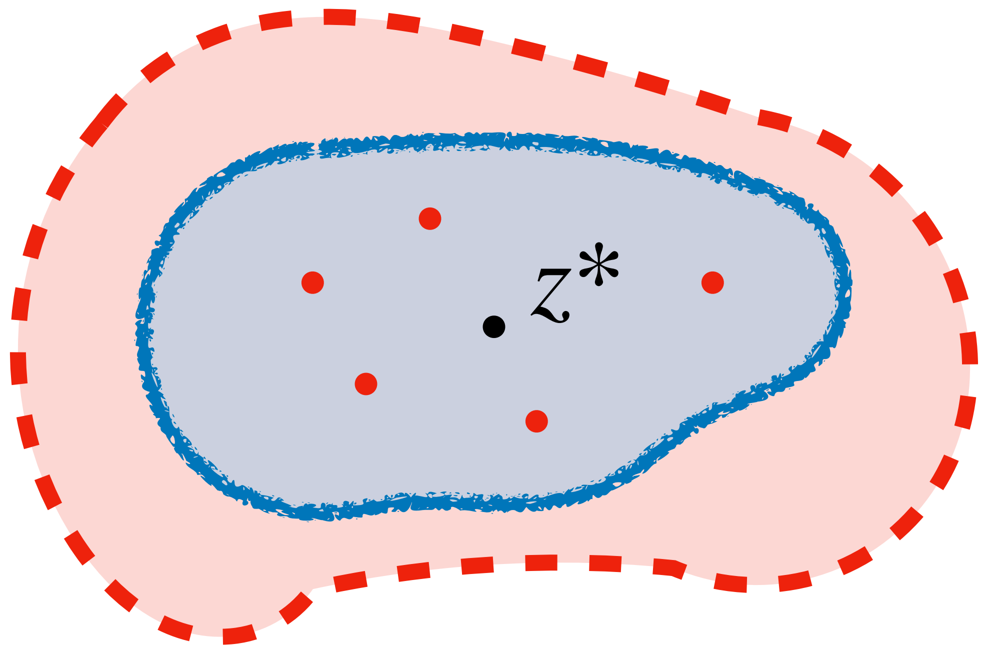

Cluster isolation using inflation

Algorithmic method

Computing R_-, R_+ and c algorithmically from a given input

Certified method

Proving the correctness of R_-, R_+ and c (not only computing them)

Counterintuitive

Inflating the multiplicity of the zero

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Two holomorphic functions $\mathscr{P}$ and $\mathscr{Q}$ on a region R with closed ∂R, if $\|\mathscr{P}(x) - \mathscr{Q}(x)\| \leq \|\mathscr{Q}(x)\|$ on ∂R, then $\mathscr{P}$ and $\mathscr{Q}$ have the same number of zeros in R°.

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Dedieu-Shub 2001, Hao-Jiang-Li-Zhi 2020, Burr-Leykin 2021

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Rouché's theorem for cluster isolation

$$\mathscr{P} = \begin{Bmatrix} x_1^2 + x_2^3 \\ x_2^2 + x_1^3 \end{Bmatrix} \text{ and } \mathscr{Q} = \begin{Bmatrix} x_1^2 \\ x_2^2 \end{Bmatrix}.$$

- $\mathscr{Q}$ has a zero at the origin of multiplicity 4.
- $\|\mathscr{P}(x) - \mathscr{Q}(x)\| \leq \|\mathscr{Q}(x)\|$ on $\mathbb{S}_\varepsilon$ (an ε-sphere) for $0 < \varepsilon \leq 1$.

It confirms the multiplicity 4 at the origin of $\mathscr{P}$.

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$$\mathscr{G} = \begin{Bmatrix} 2x_1 + x_2 + x_1^2 \\ 8x_1 + 4x_2 + x_2^2 \end{Bmatrix} \text{ with } z^* = (0,0)$$

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   where $A = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

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Canceling $x_1 x_2$ term in the first equation.

Making x_1^3 as the leading term for the first equation.

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4. Replace x_2 by $x_2 - \frac{x_1^2}{5\sqrt{5}} - \frac{x_1^3}{125}$ to make

$$\mathscr{P}_{3,3} = \mathscr{C}_3 \circ D_3 = \begin{Bmatrix} x_1^3 + \frac{x_1^4}{20\sqrt{5}} + \frac{x_1^3 x_2}{20\sqrt{5}} - \frac{x_1^2 x_2^2}{8\sqrt{5}} + \frac{x_1 x_2^3}{16\sqrt{5}} + \dots \\ x_2 - \frac{x_1 x_2}{5\sqrt{5}} + \frac{x_2^2}{20\sqrt{5}} - \frac{x_1^2 x_2}{250} + \frac{x_1^4}{500\sqrt{5}} + \dots \end{Bmatrix}$$

Deleting some terms to **inflate** x_2 term

We call $\mathscr{P}_{3,3}$ a **pre-inflatable system**

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4. Replace x_2 by $x_2 - \frac{x_1^2}{5\sqrt{5}} - \frac{x_1^3}{125}$ to make

$$\mathscr{P}_{3,3} = \mathscr{C}_3 \circ D_3 = \begin{Bmatrix} x_1^3 + \frac{x_1^4}{20\sqrt{5}} + \frac{x_1^3 x_2}{20\sqrt{5}} - \frac{x_1^2 x_2^2}{8\sqrt{5}} + \frac{x_1 x_2^3}{16\sqrt{5}} + \dots \\ x_2 - \frac{x_1 x_2}{5\sqrt{5}} + \frac{x_2^2}{20\sqrt{5}} - \frac{x_1^2 x_2}{250} + \frac{x_1^4}{500\sqrt{5}} + \dots \end{Bmatrix}$$

Deleting some terms to **inflate** x_2 term

We call $\mathscr{P}_{3,3}$ a **pre-inflatable system**

Rouché's theorem for cluster isolation using inflation

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5. Applying the **inflation operator** S_1^3 to replace x_2 by x_2^3 to make

$$\mathscr{P} = \mathscr{P}_{3,3} \circ S_1^3 = \begin{Bmatrix} +\frac{x_1^4}{20\sqrt{5}} - \frac{x_1^5}{500} - \frac{7x_1^6}{5000\sqrt{5}} + \frac{x_1^3 x_2^3}{20\sqrt{5}} + \cdots \\ +\frac{x_1^4}{500\sqrt{5}} - \frac{x_1 x_2^3}{5\sqrt{5}} + \frac{x_1^5}{31250} - \frac{x_1^2 x_2^3}{250} + \cdots \end{Bmatrix}$$

Setting $\mathscr{Q} = \begin{Bmatrix} x_1^3 \\ x_2^3 \end{Bmatrix}$ and apply Rouché's Theorem

Rouché's theorem for cluster isolation using inflation

$$\mathscr{P}_{3,3} = \mathscr{C}_3 \circ D_3 = \begin{Bmatrix} x_1^3 + \dfrac{x_1^4}{20\sqrt{5}} + \dfrac{x_1^3 x_2}{20\sqrt{5}} - \dfrac{x_1^2 x_2^2}{8\sqrt{5}} + \dfrac{x_1 x_2^3}{16\sqrt{5}} + \dots \\ x_2 - \dfrac{x_1 x_2}{5\sqrt{5}} + \dfrac{x_2^2}{20\sqrt{5}} - \dfrac{x_1^2 x_2}{250} + \dfrac{x_1^4}{500\sqrt{5}} + \dots \end{Bmatrix}$$

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Setting $\mathscr{Q} = \begin{Bmatrix} x_1^3 \\ x_2^3 \end{Bmatrix}$ and apply Rouché's Theorem

It confirms the multiplicity 9 of $\mathscr{P}$ on $\mathbb{S}_\varepsilon$ for $0 < \varepsilon \leq 1$

Rouché's theorem for cluster isolation using inflation

Applying Rouché's theorem, $\mathscr{P}$ has a zero $z^* = (0,0)$ of multiplicity 9 inside the ball $\|x_1\|^2 + \|x_2\|^2 \leq \varepsilon$ for $0 < \varepsilon \leq 1$.

Applying the inverse of each operator, we have the region

$$\frac{1}{5}\left|x_1 - 2x_2\right|^2 + \frac{1}{5^{1/3}}\left|\left(2x_1 + x_2\right) + \frac{\left(x_1 - 2x_2\right)^2}{25} + \frac{\left(x_1 - 2x_2\right)^3}{625}\right|^{\frac{2}{3}} \leq \varepsilon^2$$

It confirms the multiplicity 3 at the origin of $\mathscr{G}$.

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It confirms the multiplicity 3 at the origin of $\mathscr{G}$.

Rouché's theorem for cluster isolation using inflation (Perturbed)

$$\hat{\mathscr{G}} = \left\{ \begin{array}{c} 2x_1 + x_2 + x_1^2 + 0.001 \\ 8x_1 + 4x_2 + x_2^2 + 0.001 \end{array} \right\} \text{ with}$$

$z^* = (-0.0001, -0.0001)$ approximating 3 zeros

Locating z^* at the origin and applying the same transformations result in

Rouché's theorem for cluster isolation using inflation (Perturbed)

$$\hat{\mathscr{G}} = \begin{Bmatrix} 2x_1 + x_2 + x_1^2 + 0.001 \\ 8x_1 + 4x_2 + x_2^2 + 0.001 \end{Bmatrix} \text{ with}$$

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Locating z^* at the origin and applying the same transformations result in

$$\begin{Bmatrix} -0.0084 + 0.0013x_1 + 0.000078x_1^2 + x_1^3 + 0.0016x_2^3 + \dots \\ -0.000022 + 0.00002x_1 + 0.00000089x_1^2 + 0.00000008x_1^3 + x_2^3 + \dots \end{Bmatrix}$$

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(cubic parts) ≥ 0.4984 on the unit circle

(degree < 3 part) < 0.009757 on the unit circle

(degree > 3 part) < 0.2746 on the unit circle

Setting the cubic part by $\mathscr{Q}$ and apply Rouché's Theorem

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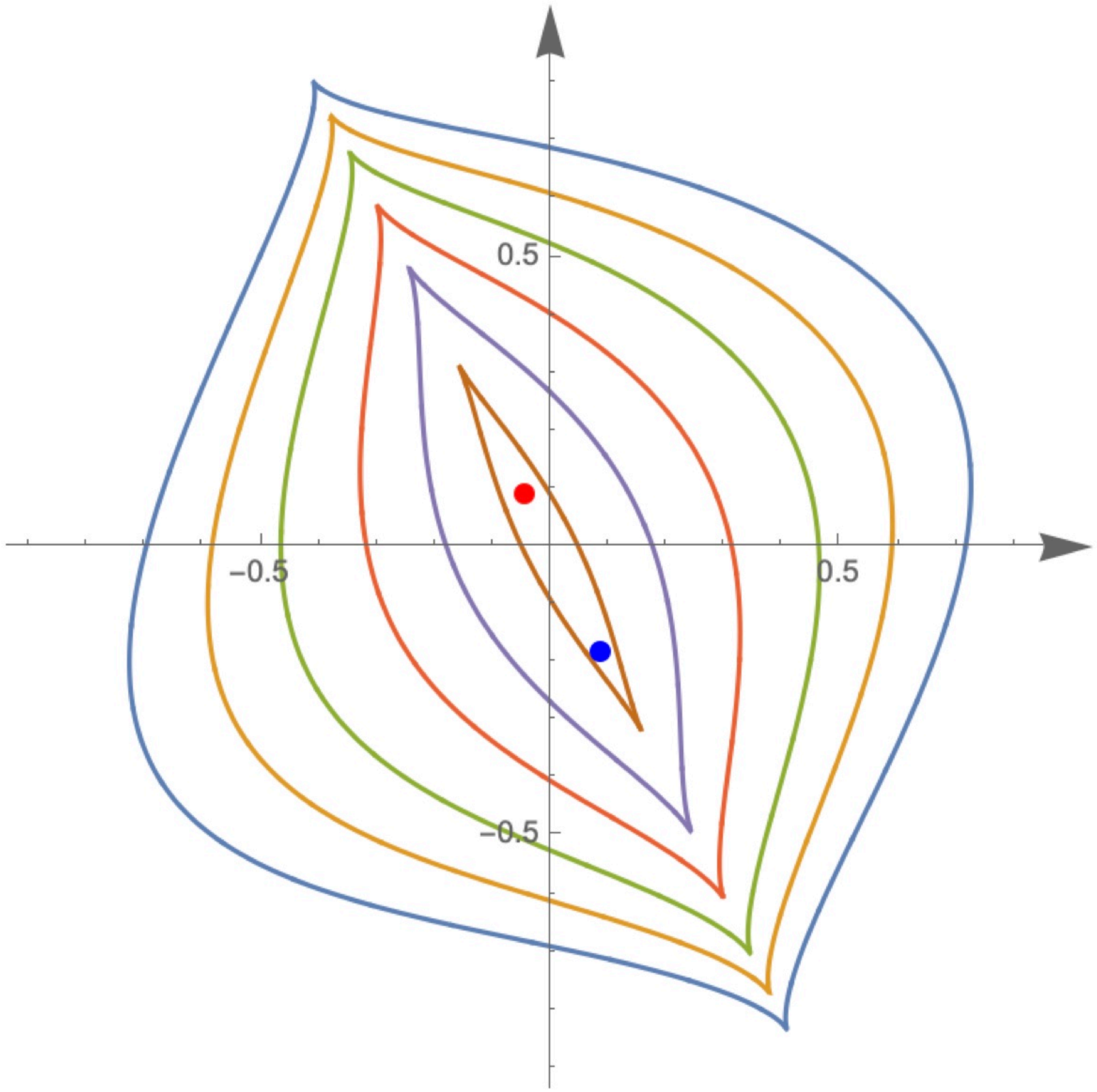
Setting the cubic part by $\mathscr{Q}$ and apply Rouché's Theorem

Applying Rouché's theorem, we may get regions isolating cluster of zeros



The red point depicts the real part of two conjugate imaginary zeros.

Algorithm 1 Pre-inflation construction

Input: A square polynomial system $\mathcal{G}$ with a singular zero z^* of breadth κ, and integers d and ℓ.

Output: A (κ, k, ℓ)-pre-inflatable system whose zero at the origin is of the same multiplicity as z^* for $\mathcal{G}$.

1: Apply an affine transformation $A : \mathbb{C}^n \to \mathbb{C}^n$ so that $A(0) = z^*$ and the kernel of the Jacobian of $\mathcal{A} = \mathcal{G} \circ A$ is spanned by the standard basis vectors $e_1, \ldots, e_\kappa$.

2: Apply a linear map $B : \mathbb{C}[x_1, \ldots, x_n]^n \to \mathbb{C}[x_1, \ldots, x_n]^n$ to construct the system $\mathcal{B} = B \circ \mathcal{A} = \{b_1, \ldots, b_n\}$ such that b_i for $i = 1, \ldots, \kappa$ do not have any linear terms and the linear form of b_i is x_i for $i > \kappa$.

3: Apply a linear map $C_k : \mathbb{C}[x_1, \ldots, x_n]^n \to \mathbb{C}[x_1, \ldots, x_n]^n$ to produce the system $\mathcal{C}_k = C_k \circ \mathcal{B} = \{c_1, \ldots, c_n\}$ such that the smallest total degree of a term with $x_{\kappa+1}, \ldots, x_n$ in $c_1, \ldots, c_\kappa$ is greater than k.

4: Apply a change of variables D_ℓ producing the system $\mathcal{P}_{k,\ell} = \mathcal{C}_k \circ D_\ell = \{p_1, \ldots, p_n\}$ such that the smallest total degree of a term in $p_{\kappa+1}, \ldots, p_n$ with only $x_1, \ldots, x_\kappa$ is greater than ℓ.

Algorithm 3 Generalized inflation for isolating clusters of zeros

Input: A square polynomial system $\mathcal{G}$ with a cluster of zeros near z^* and $d \in \mathbb{N}$.

Output: A pair of regions R_+ and R_- containing the cluster and no other zeros of $\mathcal{G}$ such that $R_- \subsetneq R_+{}^\circ$.

1: Construct a singular system $\mathcal{G}$ close to the given system.

2: Apply Algorithm 1 with parameters $k = \ell = d$ to $\mathcal{G}$ and collect the two invertible maps U and T applied to $\mathcal{G}$ as $U \circ \mathcal{G} \circ T$.

3: Compute $\mathcal{H}$ as in Equation (2).

4: Compute a lower bound M on $\mathcal{H}_d$ on the (Hermitian) unit sphere.

5: Compute an upper bound M_1 on $\mathcal{H}_{>d}/\|x\|^{d+1}$ on the unit disk.

6: Compute an upper bound M_2 on $\mathcal{H}_{<d}$ on the unit disk.

7: Compute $\varepsilon_- = \left(\frac{2M_2}{M}\right)^{1/d}$ and $\varepsilon_+ = \frac{M}{2M_1}$.

8: **if** $\varepsilon_- < \varepsilon_+$ **then**

9: Apply the inverse of $T \circ S_\kappa^d$ to the balls of radii ε_- and ε_+ to get the isolating regions R_- and R_+.

10: **end if**

Both exact and inexact cases, isolation can be done algorithmically

Regular zeros

$\mathscr{P}$: a square polynomial system such that the origin

z^* is an isolated zero with $\kappa = \dim \ker D\mathscr{P}(z^*)$

z^* is called a **regular zero of breadth** κ **and order** d

if the Hilbert series for $\langle \mathscr{P} \rangle$ at the origin is

$(1 + t + \cdots + t^{d-1})^\kappa$

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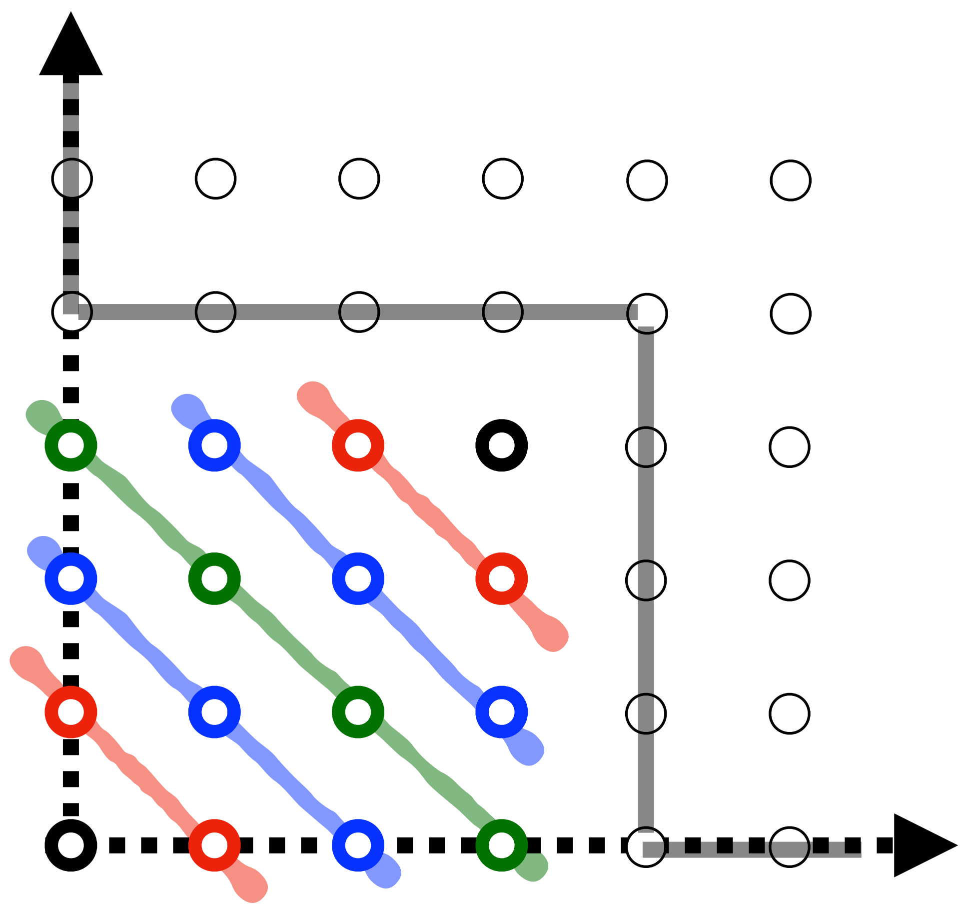

$\kappa = 2,\ d = 3$

$\mathscr{H} = (1,2,3,4,3,2,1)$

the standard monomials

form the κ-cube

Regular zeros

$\mathcal{G} = \left\{ \begin{matrix} 2x_1 + x_2 + x_1^2 \\ 8x_1 + 4x_2 + x_2^2 \end{matrix} \right\}$ with $z^* = (0,0)$ has the Hilbert series

$1 + t + t^2$

(A regular zero of breadth 1 and order 3)

$\mathcal{G} = \left\{ \begin{matrix} x_1x_2 - x_3^3 \\ x_2x_3 - x_1^3 \\ x_1x_3 - x_2^3 \end{matrix} \right\}$ with $z^* = (0,0,0)$ has the Hilbert series

$1 + 3t + 3t^2 + 3t^3 + t^4$

(Not a regular zero)

Regular zeros

$\mathcal{G} = \begin{Bmatrix} 2x_1 + x_2 + x_1^2 \\ 8x_1 + 4x_2 + x_2^2 \end{Bmatrix}$ with $z^* = (0,0)$ has the Hilbert series

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(Not a regular zero)

Regular zeros

Theorem (Burr-L.-Leykin) Let $\mathscr{G}$ be a square system in n variables with a regular zero of breadth κ and order d at z^*. Then, there is a locally invertible transformation to a pre-inflatable system $\mathscr{P} = \{p_1, \ldots, p_n\}$ such that

(1) The initial degree of each p_i is equal to d for $1 \leq i \leq \kappa$,

(2) The initial forms of $p_1, \ldots, p_\kappa$ do not vanish on the unit sphere in $x_1, \ldots, x_\kappa$, and

(3) The initial form of p_i is x_i for $\kappa + 1 \leq i \leq n$.

Irregular systems

$$\mathcal{G} = \left\{ \begin{array}{c} x_1x_2 - x_3^3 \\ x_2x_3 - x_1^3 \\ x_1x_3 - x_2^3 \end{array} \right\} \text{ with } z^* = (0,0,0)$$

From a local Gröbner basis calculation, there are three basis elements with pure powers initial terms:

$$\left\{ \begin{array}{c} x_2x_3 - x_1^3 \\ x_1x_3 - x_2^3 \\ x_1x_2 - x_3^3 \\ x_2^4 - x_3^4 \\ x_1^4 - x_2^4 \\ x_3^5 - x_1^3x_2^3 \end{array} \right\}$$

Irregular systems

$$\mathcal{G} = \left\{ \begin{matrix} x_1x_2 - x_3^3 \\ x_2x_3 - x_1^3 \\ x_1x_3 - x_2^3 \end{matrix} \right\} \text{ with } z^* = (0,0,0)$$

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Irregular systems

$$\begin{Bmatrix} x_2^4 - x_3^4 \\ x_1^4 - x_2^4 \\ x_3^5 - x_1^3 x_2^3 \end{Bmatrix}$$

Make this into a system with the same degree for initial terms:

$$\mathscr{P} = \begin{Bmatrix} x_2^5 - x_2 x_3^4 \\ x_1^5 - x_1 x_2^4 \\ x_3^5 - x_1^3 x_2^3 \end{Bmatrix}$$

Setting $\mathscr{Q} = \begin{Bmatrix} x_2^5 \\ x_1^5 \\ x_3^5 \end{Bmatrix}$ and apply Rouché's Theorem

Irregular systems

$$\mathscr{P} = \begin{Bmatrix} x_2^5 - x_2x_3^4 \\ x_1^5 - x_1x_2^4 \\ x_3^5 - x_1^3x_2^3 \end{Bmatrix}$$

1. This can be obtained by multiplying the equations of $\mathscr{G}$ by the matrix $T = \begin{pmatrix} x_2x_3 & 0 & -x_2^2 \\ 0 & -x_1^2 & x_1x_2 \\ -x_3^2 & x_2^3 & x_2x_3 \end{pmatrix}$
2. The singular zero has multiplicity 80 which is larger than the actual multiplicity 11.

Irregular systems

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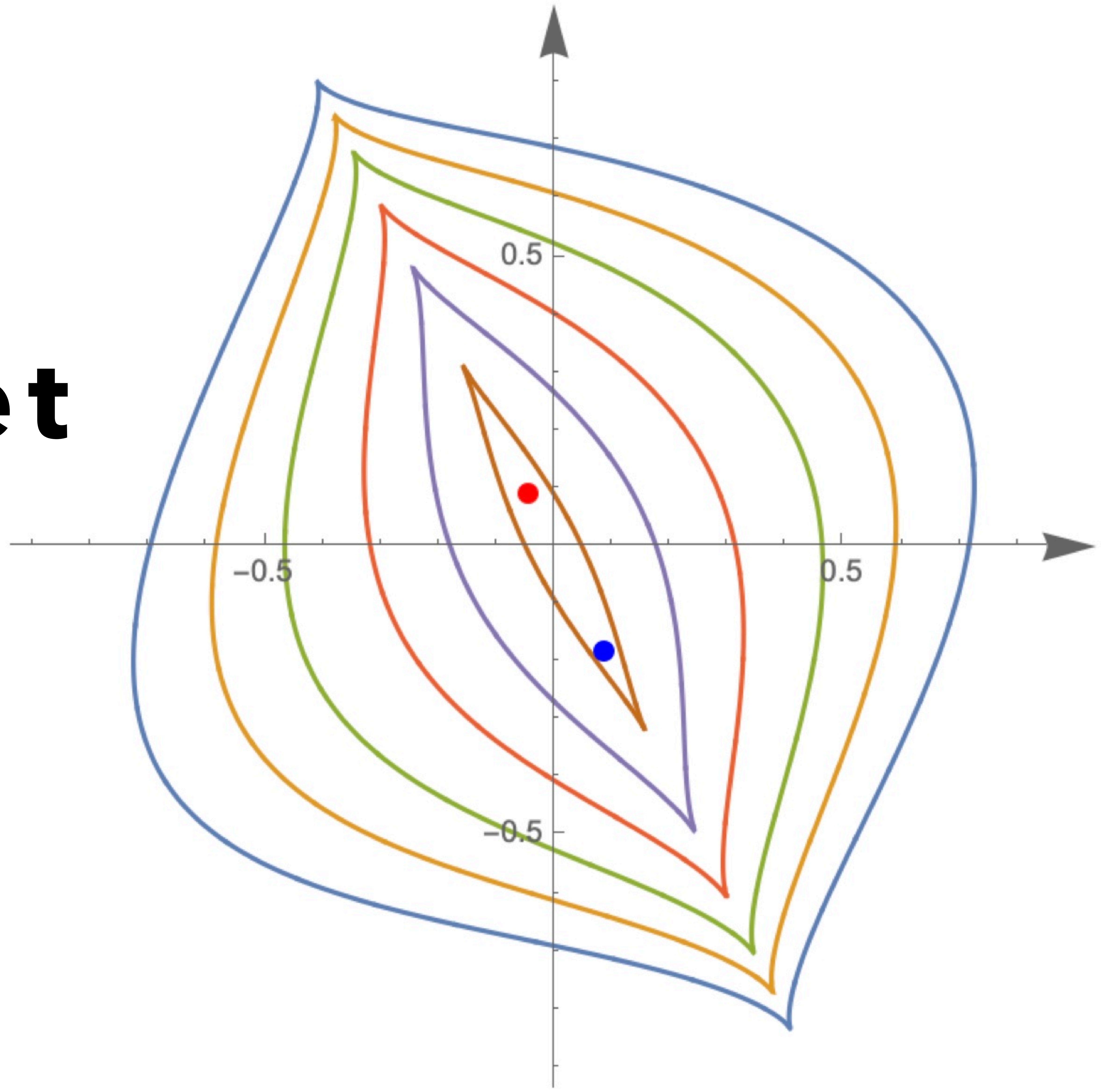
2. The singular zero has multiplicity 80 which is larger than the actual multiplicity 11.
3. A more systematic way to deal with irregular systems will be a future problem to pursue.

Takk for din oppmerksomhet

Thank you for your attention!



(https://arxiv.org/abs/2302.04776)