3.5. Group Homomorphism

Recall group isomorphism

G, H : groups.

$T: G \longrightarrow H$

~~① 1-1~~ ~~②~~ onto

$\boxed{③ \ T(ab) = T(a)T(b) \quad \text{preserving operations.}}$

Definition G, H: groups

$T: G \to H$ is a (group) homomorphism

if $T(xy) = T(x)T(y)$.

Example) ① $(\mathbb{Z}, +)$ & $(\mathbb{Z}_n, +)$

Define $T(x) = x \pmod n$

Then, $T: \mathbb{Z} \to \mathbb{Z}_n$ is a homomorphism.

(not an isomorphism
$T(n) = T(2n) = [0]$)

② S_n for $n \geq 2$, $(\{-1, +1\}, \text{multiplication})$
$(-1 \cdot -1 = 1, \ -1 \cdot 1 = -1)$

Define $\text{sgn}: S_n \to \{-1, +1\}$

with $\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$

then sgn is a homomorphism.

Definition G, H : groups, $T: G \to H$: a homomorphism.

The kernel of T is

$$\ker T = \{ g \in G \mid T(g) = e' \} (\subseteq G)$$

where e' is the identity H.

Example) ① $T: \mathbb{Z} \to \mathbb{Z}_n$

$x \mapsto x \pmod{n}$

then $T(nx) = 0 \pmod{n}$

$\Rightarrow \ker T = n\mathbb{Z}$.

② $\text{sgn}: S_n \to \{-1, +1\}$

$\Rightarrow \ker(\text{sgn}) = \{\text{even permutations}\} = A_n$.

Lemma G, H: groups, e : the identity of G

e': the identity of H.

$T: G \to H$: a homomorphism.

$\Rightarrow$ ① $T(e) = e'$ ② $T(x^{-1}) = T(x)^{-1}$.

proof) ① $T(x) = T(ex) = T(e)T(x) \Rightarrow e' = T(e)$.

(by cancelling $T(x)$)

② $T(x^{-1})T(x) = T(x^{-1}x) = T(e) = e'$

$\Rightarrow T(x)^{-1} = T(x^{-1})$

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Lemma// $T: G \to H$: a homomorphism.

T is an isomorphism if and only if ① T is onto

② $\ker T = \{e\}$.

(ker T is a trivial subgroup)

proof) It is enough to show that

T is injective if and only if $\ker T = \{e\}$.

If T is injective, then $\ker T = \{e\}$.

If $\ker T = \{e\}$, then if $T(x) = T(y)$.

$$\Rightarrow T(x)T(y)^{-1} = T(x)T(y^{-1}) = T(xy^{-1}) = e'$$

$$\Rightarrow xy^{-1} = e \Rightarrow x = y$$ //

Lemma// $T: G \to H$: a homomorphism.

$\Rightarrow \ker T \trianglelefteq G$.

proof) ① ($\ker T \leq G$)

Take any $x, y \in \ker T$. Then $T(x) = T(y) = e'$

Then, $T(xy^{-1}) = T(x)T(y^{-1}) = T(x)T(y)^{-1} = e' \cdot e' = e'$

$\Rightarrow xy^{-1} \in \ker T$. $\Rightarrow$ By 1-step subgroup test $\ker T \leq G$.

② ($a^{-1}(\ker T)a = \ker T$ for any $a \in G$).

(1) For $x \in \ker T$,

$T(a^{-1}xa) = T(a)^{-1}T(x)T(a) = T(a)^{-1}e'T(a) = e'$

$\Rightarrow a^{-1}xa \in \ker T$. $\Rightarrow a^{-1}(\ker T)a \leq \ker T$

(2) Likewise, $axa^{-1} \in \ker T$ if $x \in \ker T$.

$\Rightarrow x \in a^{-1}(\ker T)a$

$\Rightarrow \ker T \leq a^{-1}(\ker T)a$. //

Example) $(\mathbb{R}, +)$, let $T = \{x = a+bi \in \mathbb{C} \mid |x| = \sqrt{a^2+b^2} = 1\}$.

consider $(T, \cdot)$.

Define $f : \mathbb{R} \longrightarrow T$

$x \longmapsto e^{2\pi i x} \underbrace{(= \cos(2\pi x) + i\sin(2\pi x))}_{\text{Euler's formula.}}$

Then, $f(x+y) = e^{2\pi i(x+y)} = e^{2\pi i x} \cdot e^{2\pi i y} = f(x)f(y)$.

$\Rightarrow f$: a homomorphism.

Example) $\mathcal{F} = \{$collection of functions $F : \mathbb{R} \to \mathbb{R}\}$.

$\mathcal{D} = \{F : \mathbb{R} \to \mathbb{R} \mid F$ is differentiable in $\mathbb{R}\}$.

$\Rightarrow (\mathcal{F}, +)$, $(\mathcal{D}, +)$ are groups.

$(f+g)(x) = f(x) + g(x)$

ex) $f(x) = x^2$, $g(x) = 2x$ $\Rightarrow (f+g)(x) = x^2+2x$
$= f(x)+g(x)$

Define $T : \mathcal{D} \to \mathcal{F}$

$f(x) \mapsto \frac{df}{dx}$

Then T is a homomorphism

since $\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$.

$\ker T = \{$ constant functions $\}$.

ex) $f(x) = 2$. $\Rightarrow \frac{df}{dx} = 0$.