§12.3 Dot Product.

* ~~Dot product.~~

$\vec{v} = \langle v_1, v_2, v_3 \rangle$ & $\vec{w} = \langle w_1, w_2, w_3 \rangle$

$\Rightarrow \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

◎ properties.

① $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$

② $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (commutative)

③ $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$

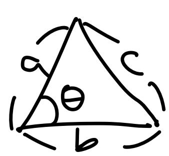
④ $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

⑤ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

* Dot product & angle.

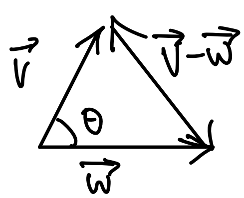
$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$, $0 \le \theta \le \pi$

Recall,



$c^2 = a^2 + b^2 - 2ab\cos\theta$

(The law of cosine)



$\Rightarrow \|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$

$\quad \parallel$

$(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \|\vec{v}\|^2 - 2\vec{v} \cdot \vec{w} + \|\vec{w}\|^2$

$\Rightarrow \|\vec{v}\| \|\vec{w}\| \cos\theta = \vec{v} \cdot \vec{w}$

ex) Find an angle between $\vec{v} = \langle 3, 6, 2 \rangle$
$\vec{w} = \langle 4, 2, 4 \rangle$

① $\vec{v} \cdot \vec{w} = 12 + 12 + 8 = 32$.

② $\|\vec{v}\| = \sqrt{3^2 + 6^2 + 2^2} = \sqrt{9 + 36 + 4} = 7$.

③ $\|\vec{w}\| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{16 + 4 + 16} = 6$.

$\Rightarrow 7 \cdot 6 \cos\theta = 32 \Rightarrow \cos\theta = \frac{32}{42} = \frac{16}{21}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{16}{21}\right)$

* orthogonality.

Suppose that $\vec{v} \cdot \vec{w} = 0 = \|\vec{v}\| \|\vec{w}\| \cos\theta$

$\vec{v} \neq 0, \vec{w} \neq 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$.

(orthogonal)
$\vec{v} \perp \vec{w}$

$\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$.

ex) Determine the orthogonality of $\vec{v}$ & $\vec{w}$
and $\vec{v}$ & $\vec{u}$

$\vec{v} = \langle 2, 6, 1 \rangle$

$\vec{u} = \langle 2, -1, 1 \rangle$

$\vec{w} = \langle -4, 1, 2 \rangle$

$\vec{v} \cdot \vec{u} = 4 - 6 + 1 = -1 \neq 0$

$\vec{v} \cdot \vec{w} = -8 + 6 + 2 = 0$

$\vec{v} \perp \vec{w}$.

* Testing obtuseness & acuteness.

Recall

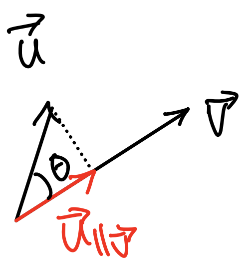
$$\begin{cases} \cos\theta \geq 0 & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ \cos\theta \leq 0 & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

$$\vec{v}\cdot\vec{w} \begin{cases} \geq 0 & \text{if } \theta \text{ is acute} \\ \leq 0 & \text{if } \theta \text{ is obtuse.} \end{cases}$$

ex) $\vec{v} = \langle 3, 1, -2 \rangle$, $\quad \vec{u} = \langle \frac{1}{2}, \frac{1}{2}, 5 \rangle$

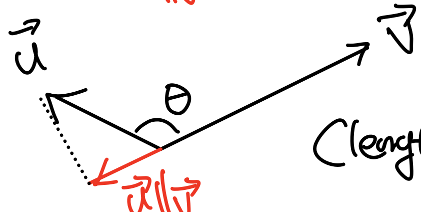
$\Rightarrow \vec{v}\cdot\vec{u} = \frac{3}{2} + \frac{1}{2} - 10 = -8 < 0 \Rightarrow$ (obtuse)

* Vector projection.



~~A projection of $\vec{u}$ along $\vec{v}$~~

$\vec{u}_{\|\vec{v}}$



(length of $\vec{u}_{\|\vec{v}}$) $= \left| \|\vec{u}\| \cos\theta \right|$

(direction of $\vec{u}_{\|\vec{v}}$) $= \begin{cases} \text{same as } \vec{v} & \text{if } \theta : \text{acute} \\ \text{opposite to } \vec{v} & \text{if } \theta : \text{obtuse.} \end{cases}$

$\Rightarrow \vec{u}_{\|\vec{v}} = \left| \|\vec{u}\| \cos\theta \right| e_{\vec{v}}$ where $e_{\vec{v}}$: a unit vector in the direction of $\vec{v}$

$= \left| \|\vec{u}\| \cos\theta \right| \frac{1}{\|\vec{v}\|} \vec{v}$

$$= \left| \frac{\|\vec{u}\|\|\vec{v}\|\cos\theta}{\|\vec{v}\|^2} \right| \vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} \right) \vec{v}$$

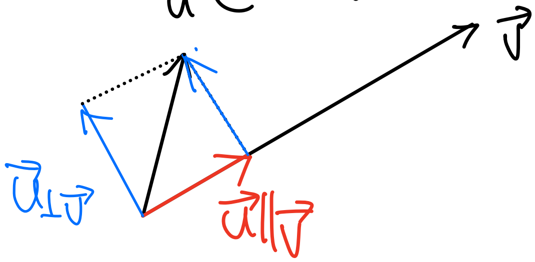
ex) Find $\vec{u}_{\parallel \vec{v}}$, $\vec{u} = \langle 5, 1, -3 \rangle$, $\vec{v} = \langle 4, 4, 2 \rangle$

① $\vec{u}\cdot\vec{v} = 20 + 4 - 6 = 18$

② $\vec{v}\cdot\vec{v} = 16 + 16 + 4 = 36$

$\Rightarrow \quad \frac{18}{36} \langle 4, 4, 2 \rangle = \langle 2, 2, 1 \rangle$

$\vec{u} \left(= \vec{u}_{\parallel \vec{v}} + \vec{u}_{\perp \vec{v}} \right)$: a decomposition of $\vec{u}$ with respect to $\vec{v}$



$\vec{u}_{\perp \vec{v}} = \vec{u} - \vec{u}_{\parallel \vec{v}}$

ex) $\vec{u} = \langle 5, 1, -3 \rangle$, $\vec{v} = \langle 4, 4, 2 \rangle$.

Find the decomposition of $\vec{u}$ w.r.t. $\vec{v}$

① $\vec{u}_{\parallel \vec{v}} = \langle 2, 2, 1 \rangle$

② $\vec{u}_{\perp \vec{v}} = \vec{u} - \vec{u}_{\parallel \vec{v}} = \langle 5, 1, -3 \rangle - \langle 2, 2, 1 \rangle$
$= \langle 3, -1, -4 \rangle$

$$\Rightarrow \quad \vec{u} = \underbrace{\langle 2, 2, 1 \rangle}_{\vec{u}_{\parallel \vec{v}}} + \underbrace{\langle 3, -1, -4 \rangle}_{\vec{u}_{\perp \vec{v}}}$$