Certifying solutions to a square system involving **analytic functions**

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Certifying (regular) Solutions

Given a compact region $I \subset \mathbb{C}^n$ (or $\mathbb{R}^n$), apply an algorithm to certify

(1) the existence

(2) the uniqueness

of a root of a system in I.

System With Analytic Functions

Consider the *error function* $\operatorname{erf}(t)$

(satisfying $\operatorname{erf}''(t) + 2t\operatorname{erf}'(t) = 0, \operatorname{erf}(0) = 0, \operatorname{erf}'(0) = \frac{2}{\sqrt{\pi}}$)

and the following square system

$$\left\{ \begin{array}{r} t_1^2 + t_2^2 = 4 \\ 2\operatorname{erf}(t_1)\operatorname{erf}(t_2) = 1 \end{array} \right\}$$

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and the following square system

$$\begin{cases} t_1^2 + t_2^2 = 4 \\ 2\,\mathrm{erf}(t_1)\,\mathrm{erf}(t_2) = 1 \end{cases}$$

rewrite as $F(t_1, t_2, t_3, t_4) = \begin{bmatrix} t_1^2 + t_2^2 - 4 \\ t_3 t_4 - \frac{1}{2} \\ t_3 - \mathrm{erf}(t_1) \\ t_4 - \mathrm{erf}(t_2) \end{bmatrix}$

Certifying solutions to a square system involving **analytic functions**

Certify an approximation x of a regular root of the following system $F : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$

$$F(x) := \begin{bmatrix} p_1(x_1, \ldots, x_{n+m}) \\ \vdots \\ p_n(x_1, \ldots, x_{n+m}) \\ x_{n+1} - g_1(x_1) \\ \vdots \\ x_{n+m} - g_m(x_m) \end{bmatrix}$$

g_i : **univariate** analytic functions (called *ingredients*)
x^* : a nonsingular (actual) root of F (i.e $F(x^*) = 0$)

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g_i : **univariate** analytic functions (called *ingredients*)
x^* : a nonsingular (actual) root of F (i.e $F(x^*) = 0$)

Given a compact region I, check the existence and uniqueness of x^* in I.

Previous Implementations

g_i - **polynomials** : Hauenstein and Sottile (2012)

g_i - **exponential functions** : Hauenstein and Levandovskyy (2017)

Both implemented in **`alphaCertified`**

Two Paradigms

Krawczyk method

combines interval arithmetic and Newton's method

Interval arithmetic

- For any arithmetic operator $\odot$, $[a,b]\odot[c,d] = \{x \odot y \mid x \in [a,b], y \in [c,d]\}$

α-Theory

certify an approximation converges to a solution quadratically

Quadratic convergence

- For $N_F(x) := x - F'(x)^{-1}F(x)$ (Newton operator),

$$\left\| N_F^k(x) - x^* \right\| \leq \left(\frac{1}{2}\right)^{2^k-1} \|x - x^*\|$$

Two Paradigms - Krawczyk method

F : a square differentiable system on $U \subset \mathbb{C}^n$

I : an interval to certify

$\square F(I) := \{F(x) \mid x \in I\}$: an interval extension of F over an interval I

y : a point in I

Y : an invertible matrix

Define the Krawczyk operator

$$K_y(I) = y - YF(y) + (Id - Y\square F'(I))(I - y)$$

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Theorem (Krawczyk 1969)**.** The following holds:

(1) If $x \in I$ is a root of F, then $x \in K_y(I)$

(2) If $K_y(I) \subset I$, then there is a root of F in I (existence)

(3) If I has a root and $\sqrt{2}\|Id - Y\square F'(I)\| < 1$, then there is root of F in I and it is unique where $\|\cdot\|$ is the maximum operator norm (uniqueness)

Two Paradigms - Krawczyk method

Over the **Real**

F : a square differentiable system on $U \subset \mathbb{R}^n$

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Two Paradigms - Krawczyk method

If $F(y)$ is hard to evaluate exactly, we use an interval $\square F(y)$ containing $F(y)$

$$\square K_y(I) = y - Y\square F(y) + (Id - Y\square F'(I))(I - y)$$

Two Paradigms - α-Theory

Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbb{C}^n$ and $N_F(x) = x - F'(x)^{-1}F(x)$.

$$\begin{aligned}
\alpha(F,x) &:= \beta(F,x)\gamma(F,x) \\
\beta(F,x) &:= \|x - N_F(x)\| = \|F'(x)^{-1}F(x)\| \\
\gamma(F,x) &:= \sup_{k\geq 2} \left\| \frac{F'(x)^{-1}F^{(k)}(x)}{k!} \right\|^{\frac{1}{k-1}}
\end{aligned}$$

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If $\alpha(F,x) < \frac{13-3\sqrt{17}}{4}$, then x converges quadratically to x^*. Also, $\|x - x^*\| \leq 2\beta(F,x)$.

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Two Paradigms - α-Theory

Theorem(*) (Burr, L., Leykin 2019)**.** *For each univariate analytic g_i, let*

(1) R_i be a positive value strictly less than the radius of convergence for g_i at x_i,

(2) M_i be an upper bound on $|g_i|$ on $\overline{D}(x_i, , R_i)$.

Then, if we let $C_i = \frac{1}{R_i}\max\left\{1, \frac{M_i}{R_i}\right\}$, then

$$\gamma(F,x) \leq \mu(F,x)\left(\frac{d^{\frac{3}{2}}}{2\|(1,x)\|} + \sum_{i=1}^{m} C_i\right).$$

where $\mu(F,x)$ is a constant depends on F and x.

Two Paradigms

Krawczyk method

$$\square K_y(I) = y - Y \square F(y) + (Id - Y \square F'(I))(I - y)$$

α-Theory

$$\gamma(F,x) \leq \mu(F,x)\left(\frac{d^{\frac{3}{2}}}{2\|(1,x)\|} + \sum_{i=1}^{m} C_i\right).$$

where $C_i = \frac{1}{R_i} \max\left\{1, \frac{M_i}{R_i}\right\}$

1) How to **evaluate** analytic functions at points (or over an interval)?
2) How to find the **radius of convergence**?

Two Oracles - *D-finite functions*

: a solution to a linear differential equation with polynomial coefficients $p_k(t) \in \mathbb{C}[t]$:

$$p_r(t)g^{(r)}(t) + \cdots + p_1(t)g'(t) + p_0(t)g(t) = 0$$

Two Oracles - *D-finite functions*

(1) van der Hoeven (1999) provides analytic continuation algorithm to approximate the value of a *D*-finite function.

(2) Mezzarobba and Salvy (2010) present algorithm to compute the *majorant series* of *D*-finite functions, which provides the radius of convergence.

Implementation : `numGfun(Maple)`,
`ore_algebra.analytic(SageMath)`

Two Oracles - *D-finite functions*

(1) Hoe gorithm to
app

We can certify a root of systems
with ***D*-finite functions!**

(2) Mez o compute
the e radius of
convergence.

Implementation : `numGfun(Maple)`,
`ore_algebra.analytic(SageMath)`

Experiments

Comparison between two methods

Consider the *error function* $\mathrm{erf}(t)$ and the following square system

$$\begin{Bmatrix} t_1^2 + t_2^2 = 4 \\ 2\,\mathrm{erf}(t_1)\,\mathrm{erf}(t_2) = 1 \end{Bmatrix} \text{ with } F(t_1, t_2, t_3, t_4) = \begin{bmatrix} t_1^2 + t_2^2 - 4 \\ t_3 t_4 - \frac{1}{2} \\ t_3 - \mathrm{erf}(t_1) \\ t_4 - \mathrm{erf}(t_2) \end{bmatrix}$$

For an approximation $t = (0.480322, 1.94147, 0.503058, 0.993961)$, we use both methods to certify this root. We round each coordinate by several decimal places to check when the methods fail.

Experiments

Comparison between two methods

decimal places	Krawczyk method	α-theory
0	fail	fail
1	pass	fail
2	pass	fail
3	pass	pass

Table 1: The Krawczyk-based method succeeds with less precision than the α-theory-based method.

Experiments

Comparison between `alphaCertified`

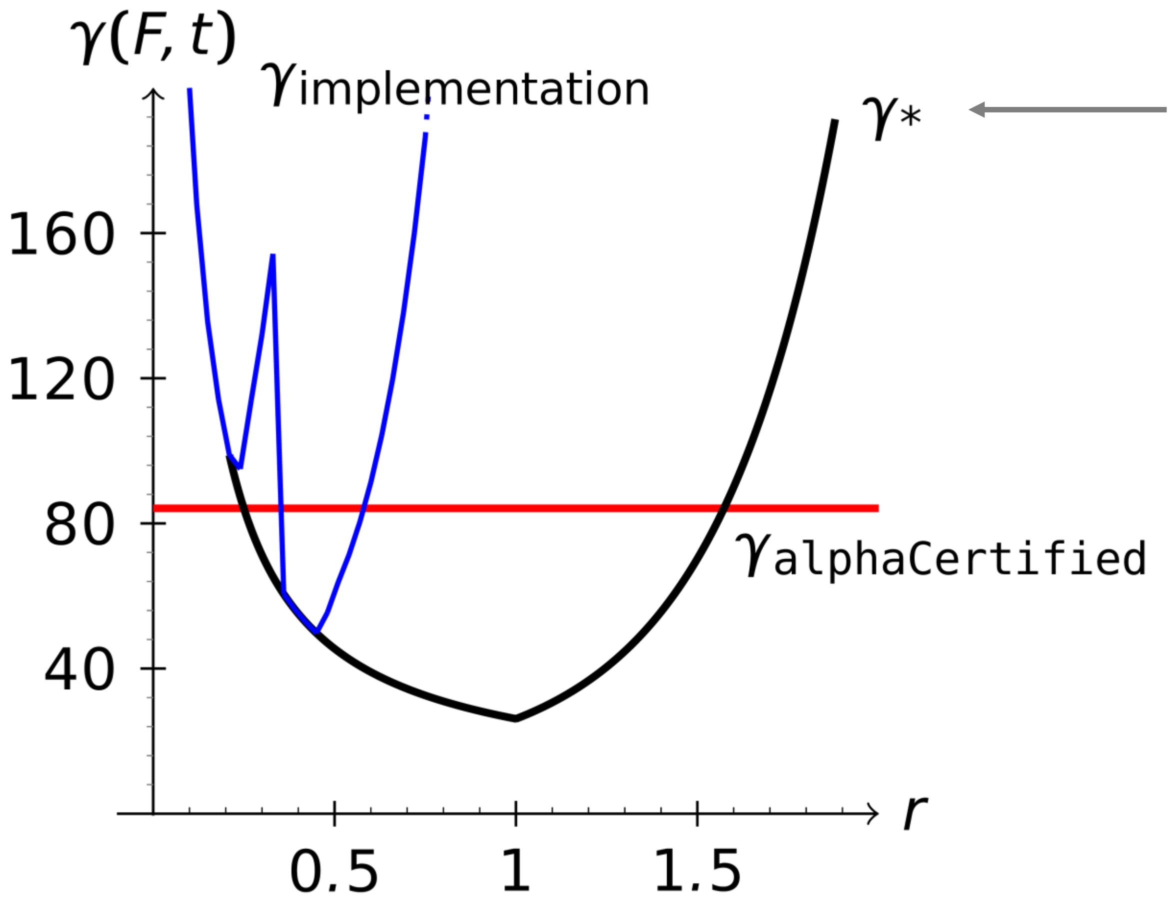
Consider the following square system:

$$\left\{e^{4t} = 0.0183\right\} \text{ with } F(t_1, t_2) = \begin{bmatrix} t_2 - 0.0183 \\ t_2 - e^{4t_1} \end{bmatrix}.$$

For an approximation $t = (-1, 0.018316)$, we compute $\gamma(F, t)$ values using `alphaCertified` and our implementation.

Experiments

Comparison between `alphaCertified`



Upper bound from Theorem (*)

$$\gamma(F,x) \leq \mu(F,x)\left(\frac{d^{\frac{3}{2}}}{2\|(1,x)\|} + \sum_{i=1}^{m} C_i\right)$$

Experiments

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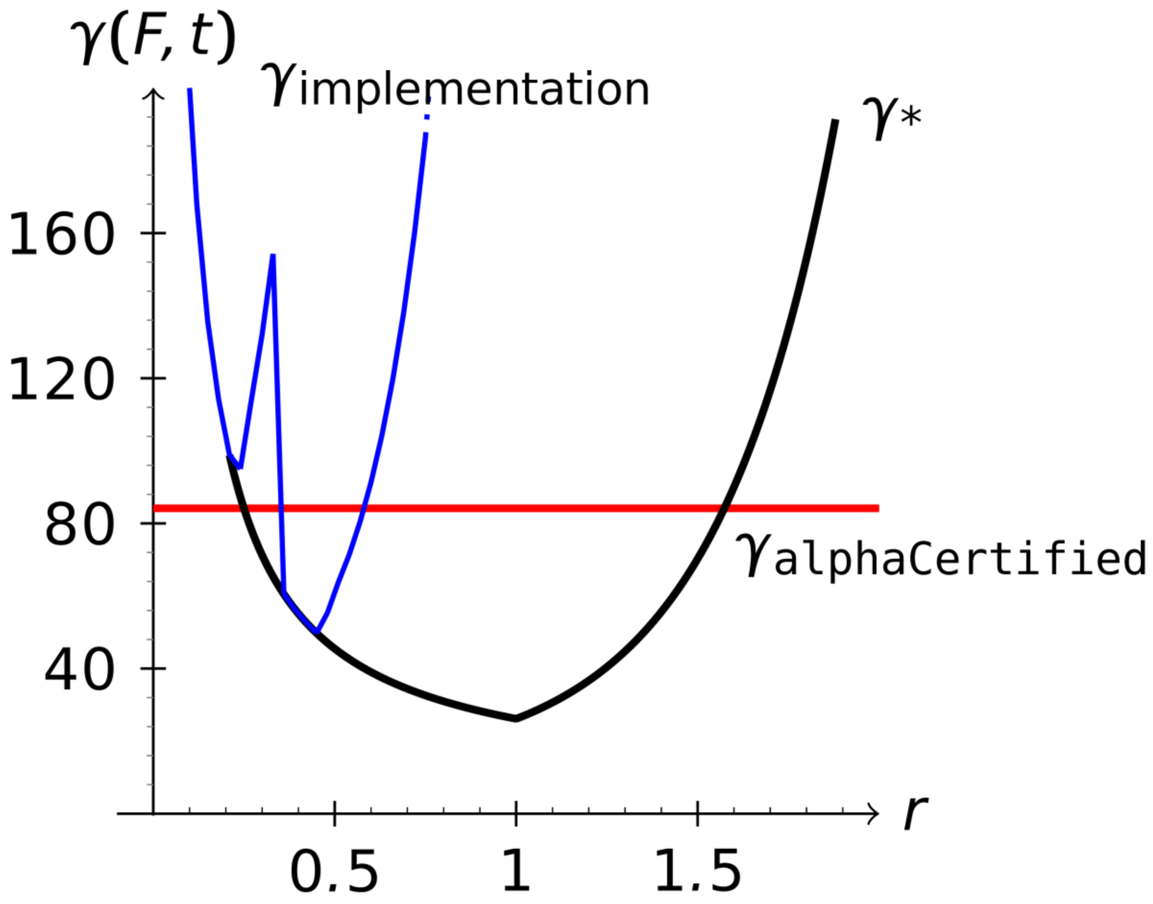
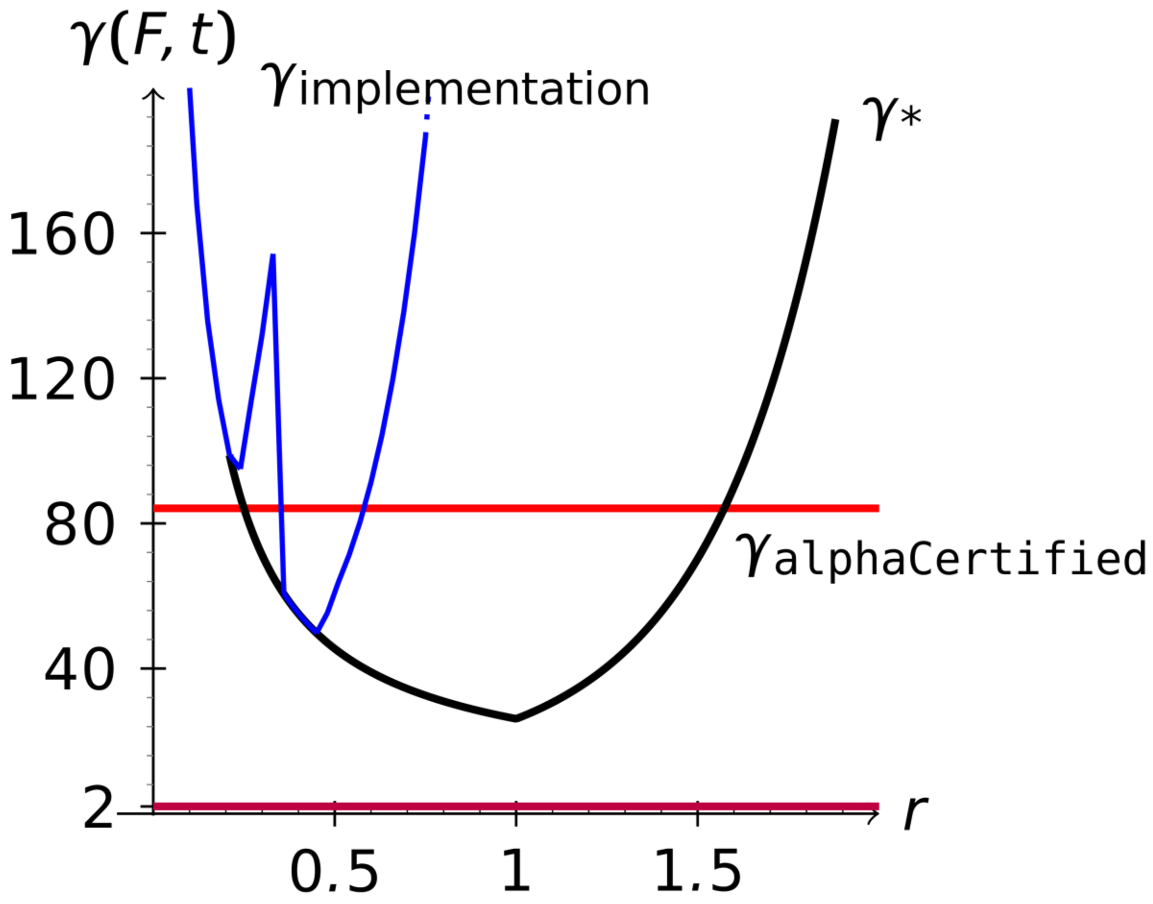


- Inexact output comes from the limitation of `ore_algebra` when it evaluates over intervals.
- For proper values of *r*, has tighter bound than `alphaCertified`
- Too big or too small *r* makes γ bigger

Experiments

Comparison between `alphaCertified`



- Actual γ value is **2 !**

Future Directions

- Oracles for other analytic functions?
  - *holonomic functions* (i.e., multivariate setting)
  - Pfaffian functions

- Certifying multiple roots?
  - Simple multiple roots **(e.g. L., Li, Zhi 2019)**

Thanks for your attention!