Certification for Roots of Systems

-- involving analytic functions

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I. Goals and Settings

: *Certify* an approximation of a regular root of the following system

$$F(x) := \begin{bmatrix} p_1(x_1,\dots,x_{n+m}) \\ \vdots \\ p_n(x_1,\dots,x_{n+m}) \\ x_{n+1} - g_1(x_1) \\ \vdots \\ x_{n+m} - g_m(x_m) \end{bmatrix}$$

x^* : a nonsingular (actual) root of F

g_i : analytic functions (called *ingredients*)

Given a compact region I, certify the existence and uniqueness of x^* in I.

Previous Works

- g_i-Polynomials : Hauenstein and Sottile (2012)
- g_i-Exponential functions : Hauenstein and Levandovskyy (2017)

II. Two Paradigms

Krawczyk Method

: combines interval arithmetic and an interval Newton's method.

: For any arithmetic operator $\odot$, $[a,b]\odot[c,d] = \{x\odot y \mid x\in[a,b], y\in[c,d]\}$

F : a square differentiable system on $U\subset\mathbb{C}^{n+m}$

$\square F(I) := \{F(x)\mid x\in I\}$: an interval extension of F over I

Y : an invertible matrix

Define the Krawczyk operator

$$K_y(I) = y - YF(y) + (Id - Y\square F'(I))(I-y)$$

Then,

(1) If $x\in I$ is a root of F, then $x\in K_y(I)$

(2) If $K_y(I)\subset I$, then there is a root of F in I

(3) If I has a root and $\sqrt{2}\|Id - Y\square F'(I)\| < 1$, then there is root of F in I and it is unique where $\|\cdot\|$ is the maximum operator norm

α-Theory

Let $x=(x_1,\dots,x_n)$ be a point in $\mathbb{C}^n$ and $N_F(x) = x - F'(x)^{-1}F(x)$.

$$\begin{aligned} \alpha(F,x) &:= \beta(F,x)\gamma(F,x) \\ \beta(F,x) &:= \|x - N_F(x)\| = \|F'(x)^{-1}F(x)\| \\ \gamma(F,x) &:= \sup_{k\geq 2}\left\|\frac{F'(x)^{-1}F^{(k)}(x)}{k!}\right\|^{\frac{1}{k-1}} \end{aligned}$$

If $\alpha(F,x) < \frac{13-3\sqrt{17}}{4}$, then x converges quadratically to x^*. Also, $\|x-x^*\|\leq 2\beta(F,x)$.

Theorem. *For each univariate analytic g_i, let*

(1) R_i be a positive value strictly less than the radius of convergence for g_i at x_i,

(2) M_i be an upper bound on $|g_i|$ on $\overline{D}(x_i,,R_i)$.

Then, if we let $C_i = \frac{1}{R_i}\max\left\{1, \frac{M_i}{R_i}\right\}$, then

$$\gamma(F,x)\leq \mu(F,x)\left(\frac{d^{\frac{3}{2}}}{2\|(1,x)\|} + \sum_{i=1}^{m} C_i\right).$$

where $\mu(F,x)$ is a constant depends on F and x.

Q. How to compute (1) and (2) ?

III. Two Oracles – for D-finite functions

D-finite function

: a solution to a linear differential equation with polynomial coefficients $p_k(t)\in\mathbb{C}[t]$:

$$p_r(t)g^{(r)}(t) + \cdots + p_1(t)g'(t) + p_0 g(t) = 0$$

(1) Mezzarobba and Salvy (2010) present an algorithm to compute the *majorant series* of D-finite functions which give the radius of convergence.
Implementation : `numGfun(Maple)`, `ore_algebra.analytic(SageMath)`

(2) Hoeven (1999) provides the analytic continuation algorithm to approximate the value of a D-finite function.
Implementation : `ore_algebra.analytic(SageMath)`

IV. Experiments

SageMath implementation is available at
`https://github.com/klee669/DfiniteComputationResults`

Comparison between α-theory and Krawczyk method

Consider the *error function* $\mathrm{erf}(t)$ and the following square system

$$\left\{\begin{aligned} t_1^2+t_2^2 &= 4 \\ 2\,\mathrm{erf}(t_1)\,\mathrm{erf}(t_2) &= 1 \end{aligned}\right\} \text{ with } F(t_1,t_2,t_3,t_4) = \begin{bmatrix} t_1^2+t_2^2-4 \\ t_3t_4 - \frac{1}{2} \\ t_3 - \mathrm{erf}(t_1) \\ t_4 - \mathrm{erf}(t_2) \end{bmatrix}$$

For an approximation $t = (0.480322, 1.94147, 0.503058, 0.993961)$, we use both methods to certify this. We round each coordinate by various decimal places to check when the methods fail.

decimal places	Krawczyk method	α-theory
0	fail	fail
1	pass	fail
2	pass	fail
3	pass	pass

Table 1: Comparison between the precision required for the Krawczyk-based and α-theory-based methods. In this case, the Krawczyk-based method succeeds with less precision than the α-theory-based method.

Comparison between `alphaCertified`

`alphaCertified` certifies a root of not only polynomial systems but also systems with exponential functions. Consider the following square system:

$$\{e^{4t} = 0.0183\} \text{ with } F(t_1,t_2) = \begin{bmatrix} t_2 - 0.0183 \\ t_2 - e^{4t_1} \end{bmatrix}.$$

For an approximation $t = (-1, 0.018316)$, we compute $\gamma(F,t)$ values using `alphaCertified` and our implementation.

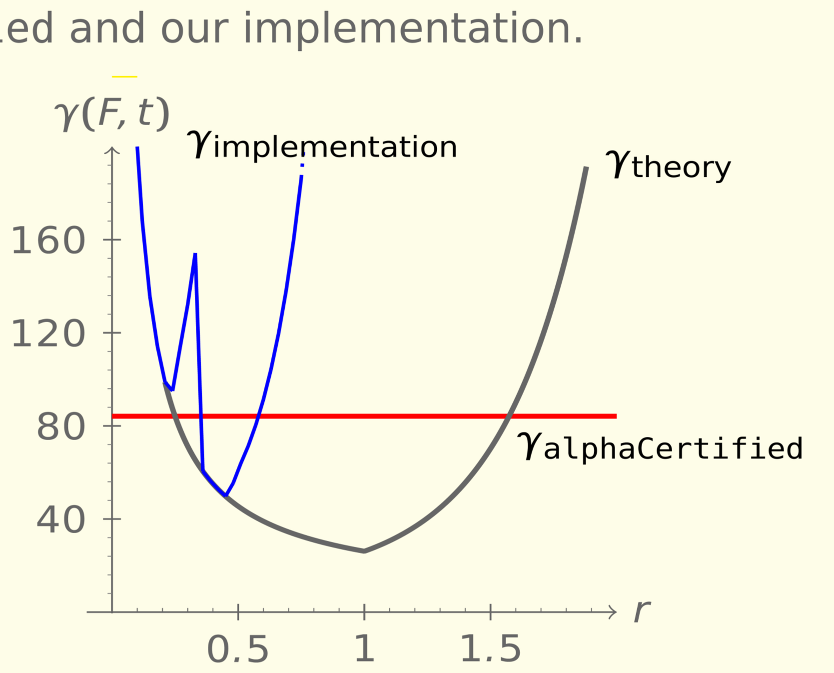

Figure 1: Comparison of computed γ values those from `alphaCertified`. For some choices of r γ_{theory} and $\gamma_{\text{implementation}}$ have lower values of γ than that of `alphaCertified`.
The implementation bounds differ from the theoretical bounds due to limitations of the `ore_algebra.analytic` package.