Polyhedral Homotopy Method for Nash Equilibrium Problem

(joint work with Xindong Tang)

Kisun Lee (UC San Diego) - kil004@ucsd.edu

AMA Colloquium Series on Young Scholars in Optimization and Data Science

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player

		2nd	
		C	D
1st	A	8	4
	B	6	7

1st player payoff

		2nd	
		C	D
1st	A	4	6
	B	5	8

2nd player payoff

		2nd	
		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player

		2nd	
		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player

		2nd	
		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player

		2nd	
		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

Discrete Math

Nash Equilibrium Problem

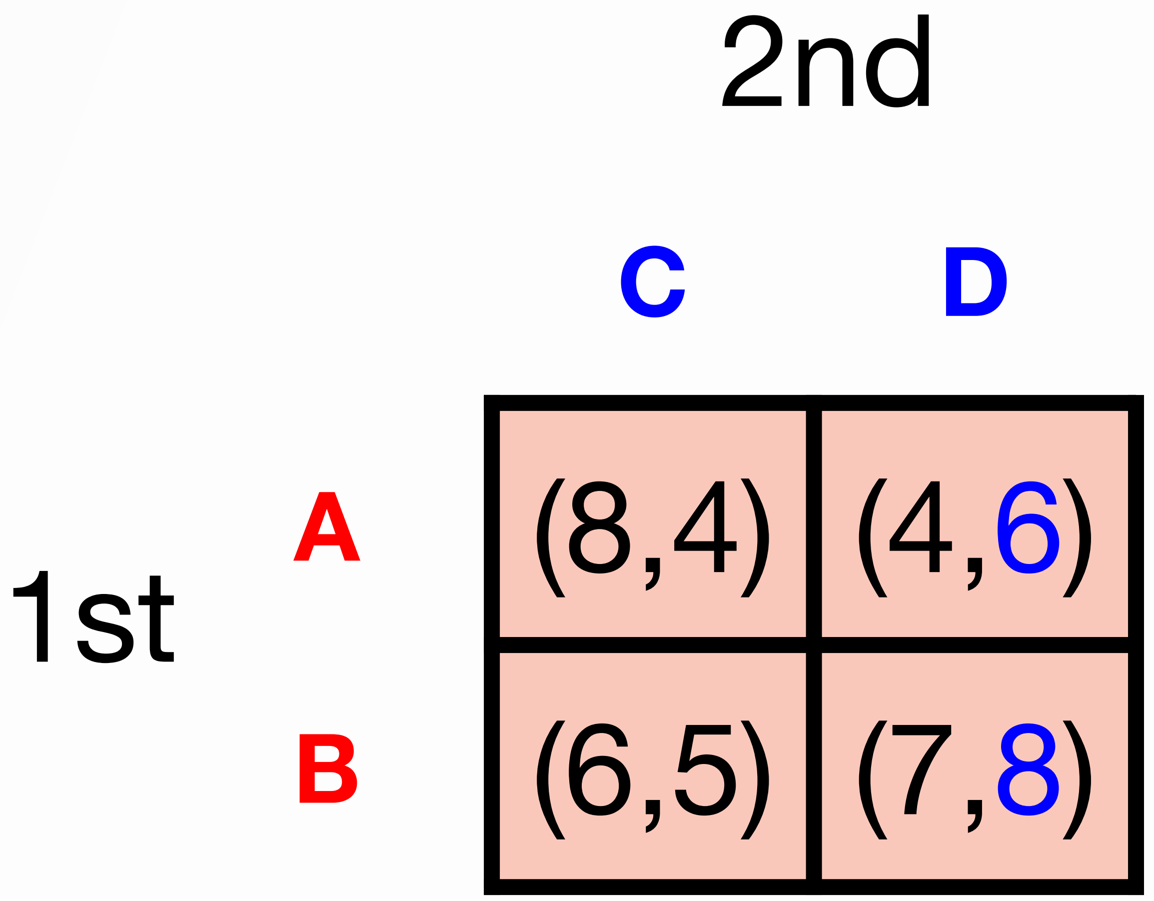
Consider a 2-player game with 2 strategies for each player

		2nd	
		C	**D**
1st	**A**	(8,4)	(4,6)
	B	(6,5)	(7,8)

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player



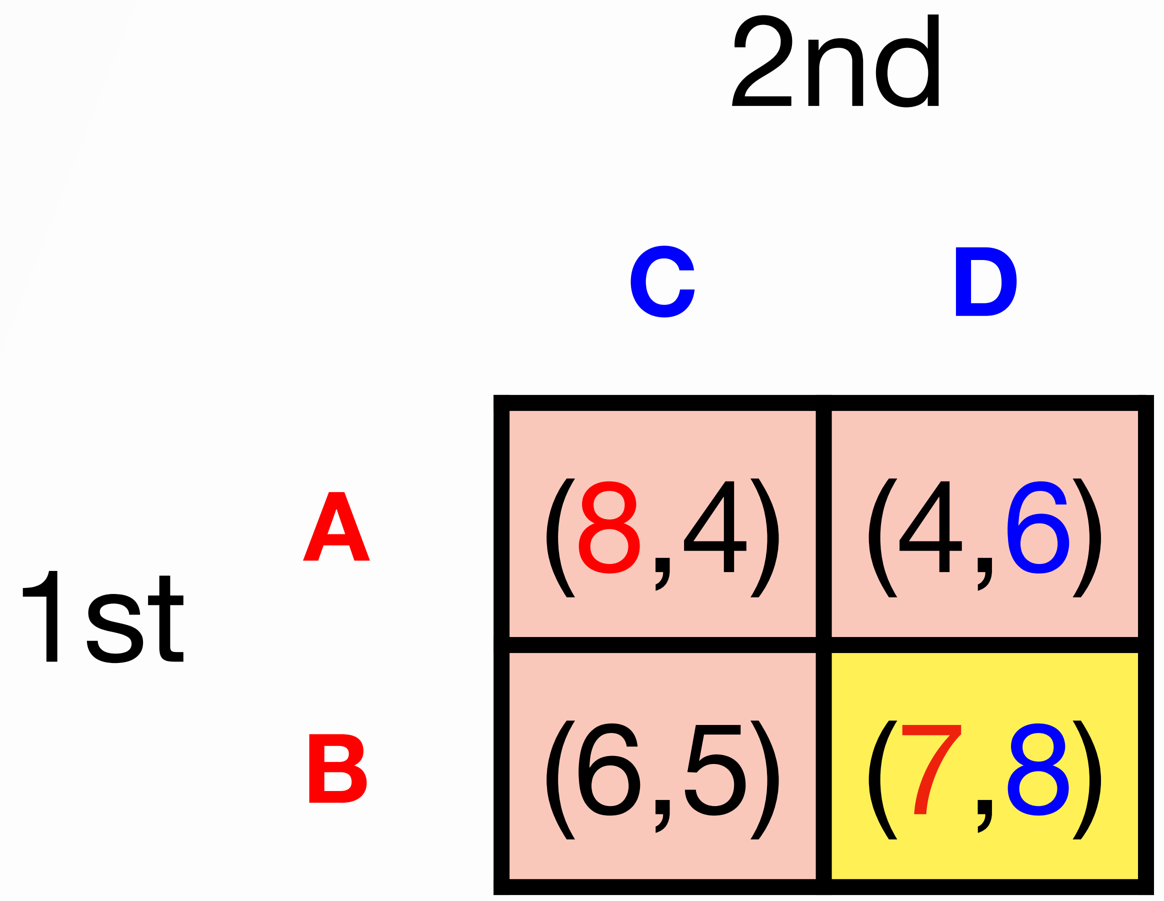
2nd

		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player



		2nd	
		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player

2nd

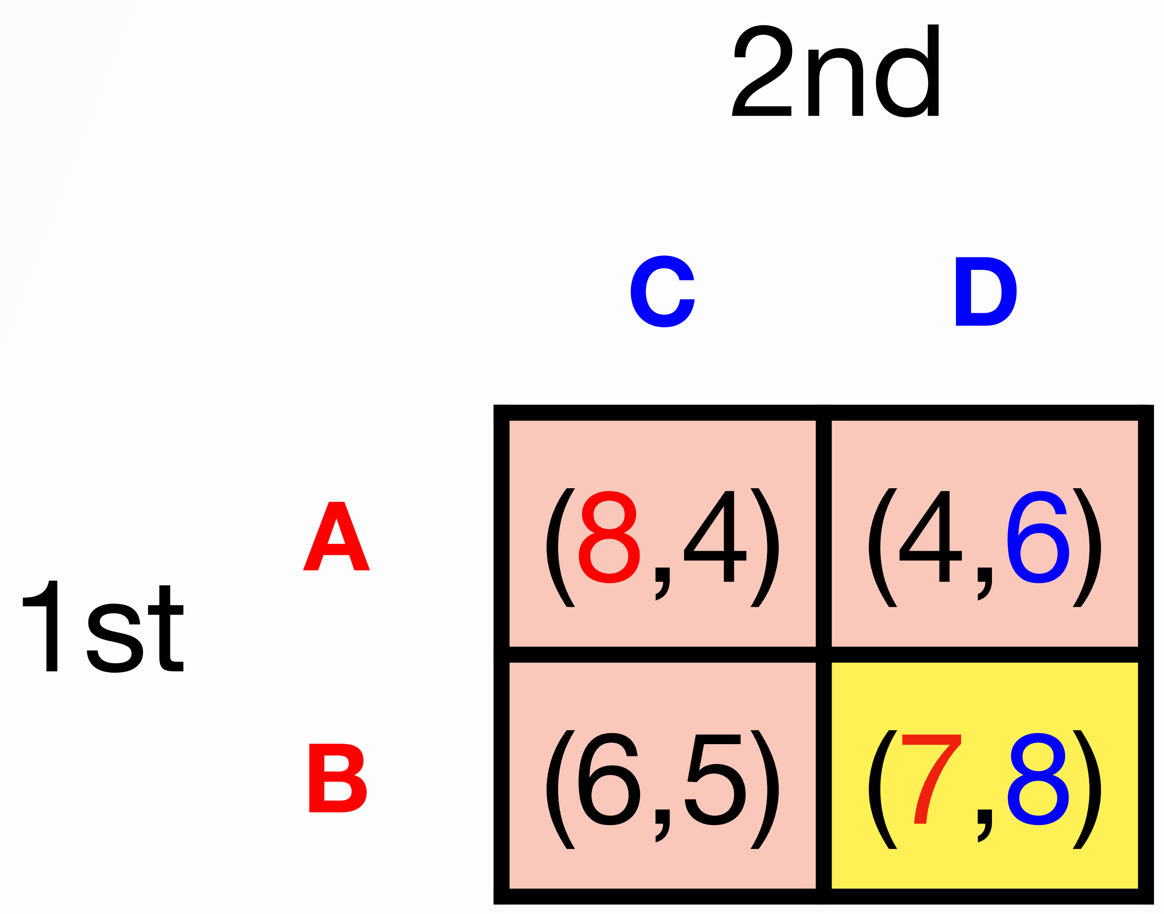
		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

(B,D) is a Nash equilibrium

Discrete Math

Nash Equilibrium Problem

Consider a 2-player game with 2 strategies for each player



		2nd	
		C	D
1st	A	(8,4)	(4,6)
	B	(6,5)	(7,8)

(B,D) is a Nash equilibrium

(A,D) is not a Nash equilibrium

Nash Equilibrium Problem

Nash equilibrium (NE)

In game theory, a state that a player can achieve the desired outcome by not changing their initial strategy.

It is a state that every player's objective is optimized for given other players' strategies.

Nash equilibrium problem

A problem finding such Nash equilibria is called the Nash equilibrium problem (NEP).

Nash Equilibrium Problem

Nash equilibrium (NE)

In game theory, a state that a player can achieve the desired outcome by not changing their initial strategy.

It is a state that every player's objective is optimized for given other players' strategies.

Nash equilibrium problem

A problem finding such Nash equilibria is called the Nash equilibrium problem (NEP).

Nash Equilibrium Problem

Nash equilibrium (NE)

In game theory, a state that a player can achieve the desired outcome by not changing their initial strategy.

It is a state that every player's objective is optimized for given other players' strategies.

Nash equilibrium problem

A problem finding such Nash equilibria is called the **Nash equilibrium problem (NEP).**

Algebra + Numerical Analysis

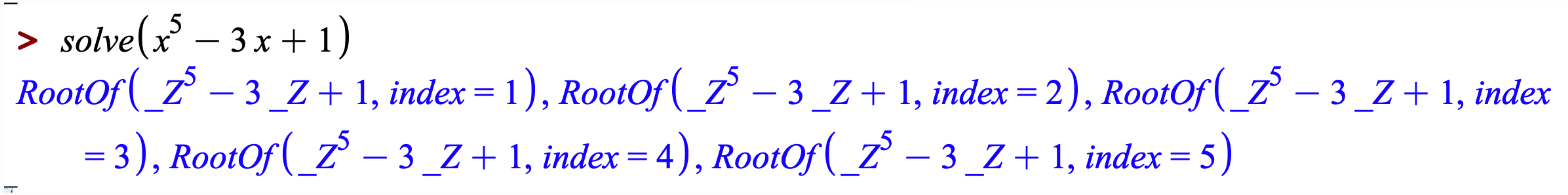
How to solve an equation?

How to find roots of $f(x) = x^5 - 3x + 1$?

Algebra + Numerical Analysis

How to solve an equation?

How to find roots of $f(x) = x^5 - 3x + 1$?



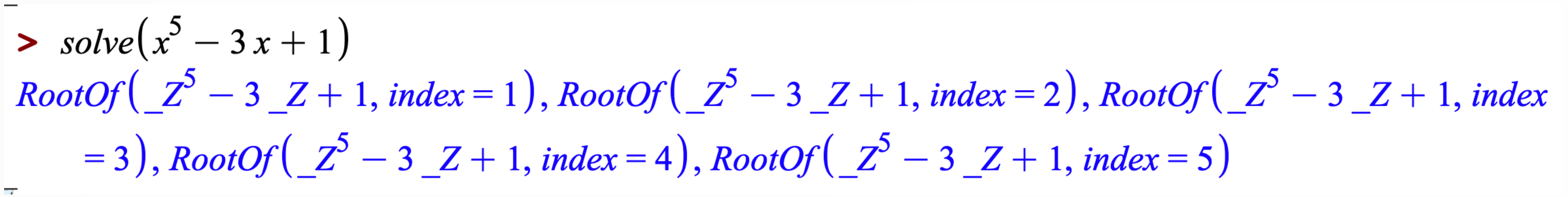
$$> \; solve\left(x^5 - 3\,x + 1\right)$$

$$RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 1\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 2\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 3\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 4\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 5\right)$$

Algebra + Numerical Analysis

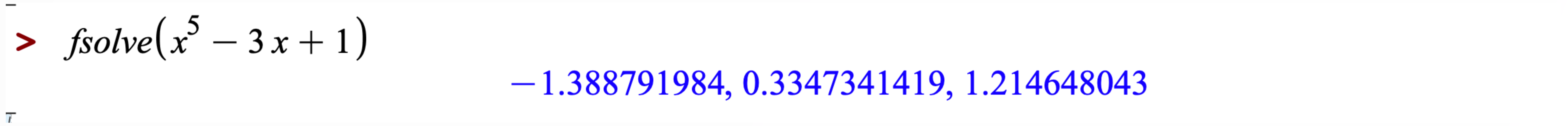
How to solve an equation?

How to find roots of $f(x) = x^5 - 3x + 1$?



> $solve\left(x^5 - 3\,x + 1\right)$

$RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 1\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 2\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 3\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 4\right), RootOf\left(\_Z^5 - 3\,\_Z + 1, index = 5\right)$



> $fsolve\left(x^5 - 3\,x + 1\right)$

$-1.388791984, 0.3347341419, 1.214648043$

Algebra + Numerical Analysis

How to solve an equation? (Geometric point of view)

How to find roots of $f(x) = x^5 - 3x + 1$?

Consider $g(x) = x^5 - 1$ (whose roots are the 5-th roots of unity $\xi_1, \ldots, \xi_5$).

Then, $H(x, t) = (1 - t)f(x) + tg(x)$ finds roots of $f(x)$ as t goes from 1 to 0.

Algebra + Numerical Analysis

How to solve an equation? (Geometric point of view)

How to find roots of $f(x) = x^5 - 3x + 1$?

Consider $g(x) = x^5 - 1$ (whose roots are the 5-th roots of unity $\xi_1, \ldots, \xi_5$).

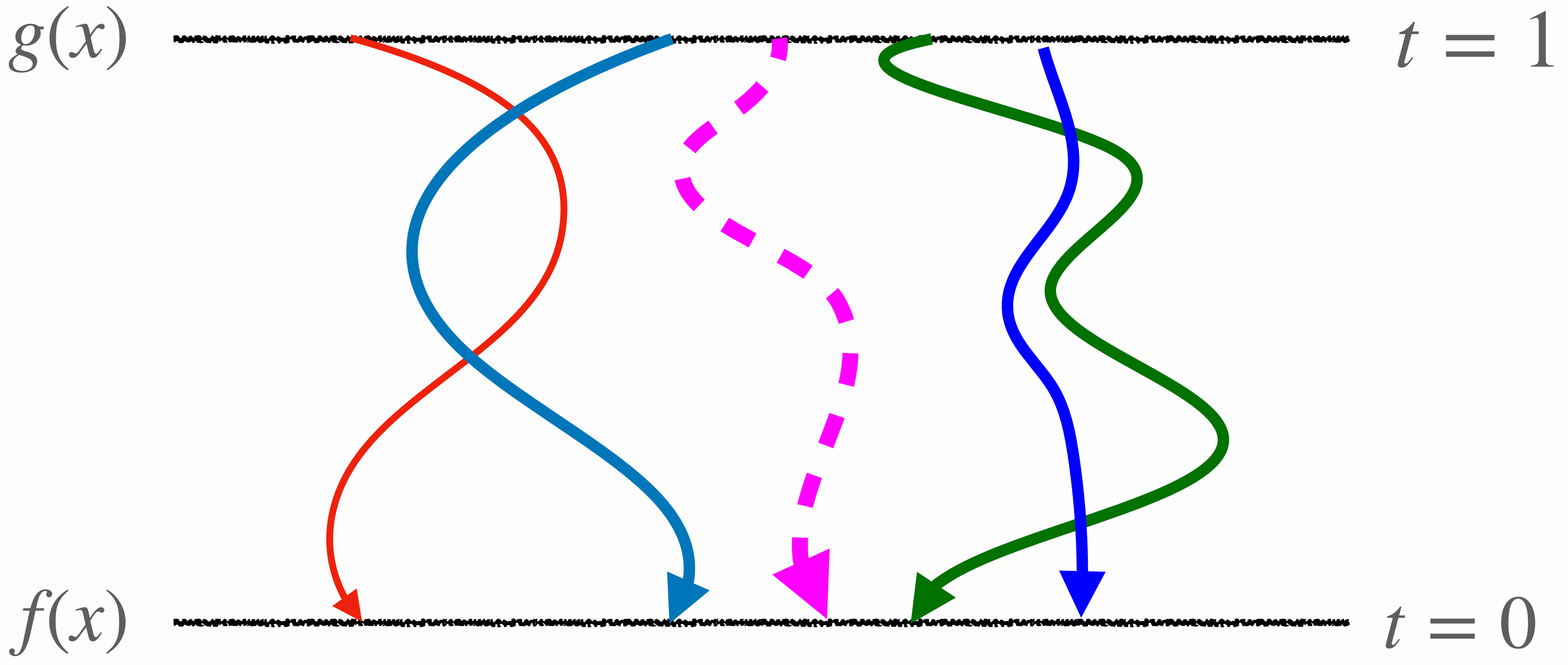
Then, $H(x, t) = (1 - t)f(x) + tg(x)$ finds roots of $f(x)$ as t goes from 1 to 0.

Algebra + Numerical Analysis

How to solve an equation? (Geometric point of view)

How to find roots of $f(x) = x^5 - 3x + 1$?

Consider $g(x) = x^5 - 1$ (whose roots are the 5-th roots of unity $\xi_1, \ldots, \xi_5$).



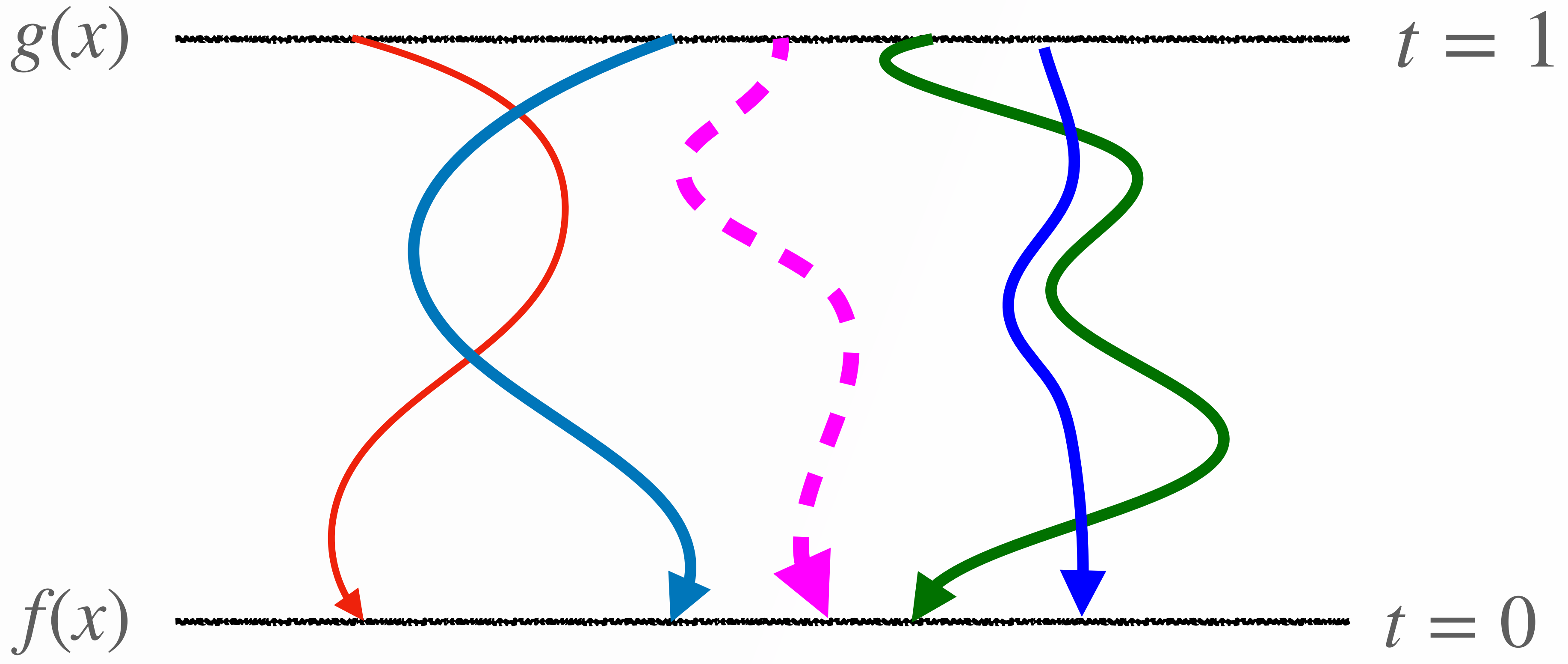
Then, $H(x, t) = (1 - t)f(x) + tg(x)$ finds roots of $f(x)$ as t goes from 1 to 0.

Algebra + Numerical Analysis

How to solve an equation? (Geometric point of view)

How to find roots of $f(x) = x^5 - 3x + 1$?

Consider $g(x) = x^5 - 1$ (whose roots are the 5-th roots of unity $\xi_1, \ldots, \xi_5$).



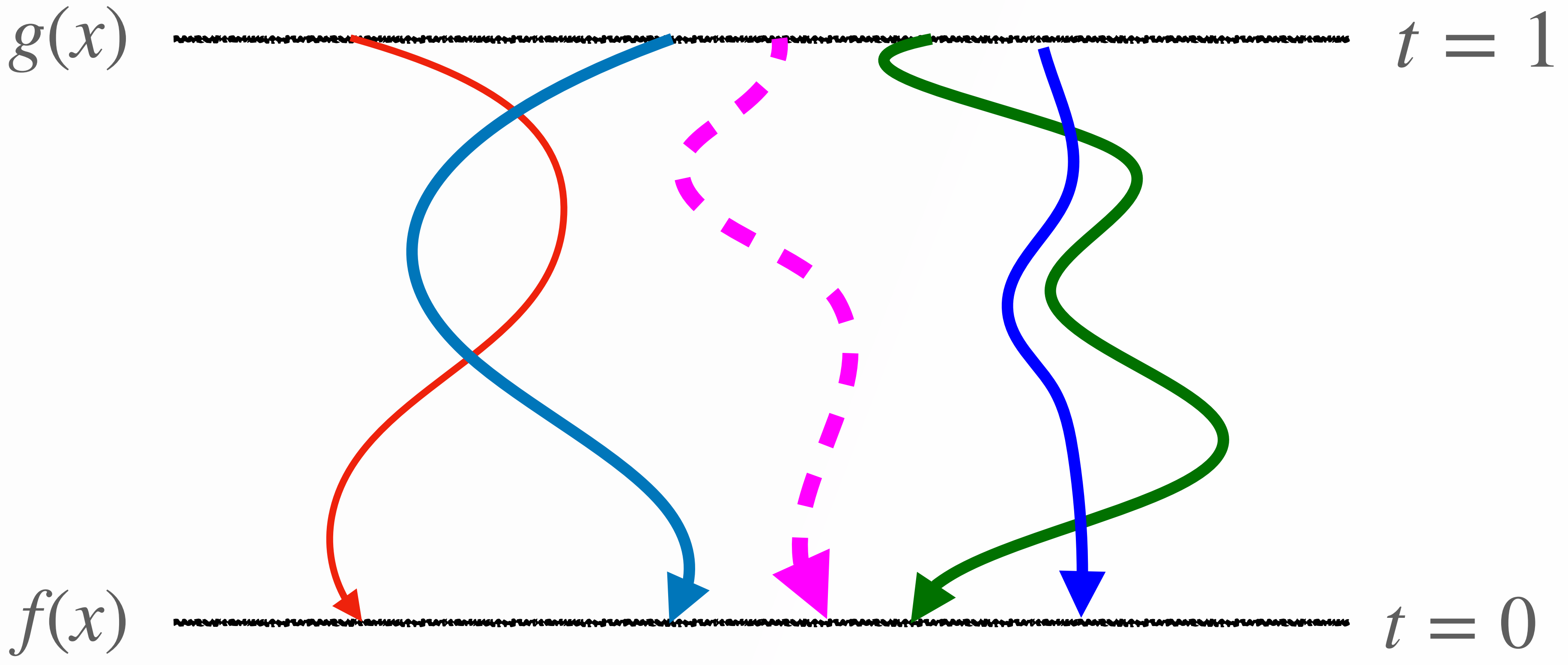
Then, $H(x, t) = (1 - t)f(x) + tg(x)$ finds roots of $f(x)$ as t goes from 1 to 0.

Algebra + Numerical Analysis

How to solve an equation? (Geometric point of view)

How to find roots of $f(x) = x^5 - 3x + 1$?

Consider $g(x) = x^5 - 1$ (whose roots are the 5-th roots of unity $\xi_1, \ldots, \xi_5$).



Then, $H(x, t) = (1 - t)f(x) + tg(x)$ finds roots of $f(x)$ as t goes from 1 to 0. **(Homotopy method)**

Nash Equilibrium Problem
+ Numerical Algebraic Geometry

Why Homotopy Method?

- Current known methods for NEP are based on optimization methods.
  - Heavily relies on the convexity of feasible sets.

- Previous works focus on finding one NE (or finding all NEs one-by-one).
  - Homotopy methods can be proper for finding all NEs at once.

Nash Equilibrium Problem

NEP as an optimization problem.

Consider N-player game.

$x_i := (x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{R}^{n_i}$

the i-th player's strategy.

$x := (x_1, \ldots, x_N) \in \mathbb{R}^{n_1+\cdots+n_N}$

a vector for all players' strategies.

$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

all strategies except i-th player's strategy.

$f_i \in \mathbb{C}[x]$

the i-th player's objective function.

$g_{i,j} \in \mathbb{C}[x]$

the i-th player's constraints $(j = 1, \ldots, m_i)$

Nash Equilibrium Problem

NEP as an optimization problem.

Consider N-player game.

$x_i := (x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{R}^{n_i}$

the i-th player's strategy.

$x := (x_1, \ldots, x_N) \in \mathbb{R}^{n_1+\cdots+n_N}$

a vector for all players' strategies.

$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

all strategies except i-th player's strategy.

$f_i \in \mathbb{C}[x]$

the i-th player's objective function.

$g_{i,j} \in \mathbb{C}[x]$

the i-th player's constraints $(j = 1, \ldots, m_i)$

Nash Equilibrium Problem

NEP as an optimization problem.

Consider N-player game.

$x_i := (x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{R}^{n_i}$

the i-th player's strategy.

$x := (x_1, \ldots, x_N) \in \mathbb{R}^{n_1+\cdots+n_N}$

a vector for all players' strategies.

$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

all strategies except i-th player's strategy.

$f_i \in \mathbb{C}[x]$

the i-th player's objective function.

$g_{i,j} \in \mathbb{C}[x]$

the i-th player's constraints $(j = 1, \ldots, m_i)$

Nash Equilibrium Problem

NEP as an optimization problem.

Consider N-player game.

$x_i := (x_{i,1}, \dots, x_{i,n_i}) \in \mathbb{R}^{n_i}$

the i-th player's strategy.

$x := (x_1, \dots, x_N) \in \mathbb{R}^{n_1+\cdots+n_N}$

a vector for all players' strategies.

$x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$

all strategies except i-th player's strategy.

$f_i \in \mathbb{C}[x]$

the i-th player's objective function.

$g_{i,j} \in \mathbb{C}[x]$

the i-th player's constraints $(j = 1, \dots, m_i)$

Nash Equilibrium Problem

NEP as an optimization problem.

Consider N-player game.

$x_i := (x_{i,1}, \dots, x_{i,n_i}) \in \mathbb{R}^{n_i}$

the i-th player's strategy.

$x := (x_1, \dots, x_N) \in \mathbb{R}^{n_1+\cdots+n_N}$

a vector for all players' strategies.

$x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$

all strategies except i-th player's strategy.

$f_i \in \mathbb{C}[x]$

the i-th player's objective function.

$g_{i,j} \in \mathbb{C}[x]$

the i-th player's constraints $(j = 1, \dots, m_i)$

Nash Equilibrium Problem

NEP as an optimization problem.

Consider N-player game.

$x_i := (x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{R}^{n_i}$

the i-th player's strategy.

$x := (x_1, \ldots, x_N) \in \mathbb{R}^{n_1+\cdots+n_N}$

a vector for all players' strategies.

$x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

all strategies except i-th player's strategy.

$f_i \in \mathbb{C}[x]$

the i-th player's objective function.

$g_{i,j} \in \mathbb{C}[x]$

the i-th player's constraints $(j = 1,\ldots, m_i)$

Nash Equilibrium Problem

NEP as an optimization problem.

Find a tuple $u = (u_1, \ldots, u_N)$ such that u_i is a optimizer of the i-th player's optimization :

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) = 0 \quad \text{if } j \in \mathscr{E}_i \\ & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \geq 0 \quad \text{if } j \in \mathscr{I}_i \end{cases}$$

where $\mathscr{E}_i$ and $\mathscr{I}_i$ are sets of indices for equality constraints and inequality constraints respectively.

Nash Equilibrium Problem

NEP as an optimization problem.

Find a tuple $u = (u_1, \ldots, u_N)$ such that u_i is a optimizer of the i-th player's optimization :

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) = 0 \quad \text{if } j \in \mathscr{E}_i \\ & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \geq 0 \quad \text{if } j \in \mathscr{I}_i \end{cases}$$

where $\mathscr{E}_i$ and $\mathscr{I}_i$ are sets of indices for equality constraints and inequality constraints respectively.

Nash Equilibrium Problem

NEP as an optimization problem.

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) = 0 \quad \text{if } j \in \mathcal{E}_i \\ & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \geq 0 \quad \text{if } j \in \mathcal{I}_i \end{cases}$$

$$X_i := \{x_i \in \mathbb{R}^{n_i} \mid g_{i,j}(x_i) = 0, \quad g_{i,j}(x_i) \geq 0\}$$

the feasible set of F_i

regular NEP

the feasible set X_i doesn't depend on x_{-i}.

generalized NEP (GNEP)

the feasible set X_i depends on x_{-i}.

GNEP of polynomials (GNEPP)

all f_i and $g_{i,j}$ are polynomials.

Nash Equilibrium Problem

NEP as an optimization problem.

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) = 0 \quad \text{if } j \in \mathscr{E}_i \\ & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \geq 0 \quad \text{if } j \in \mathscr{I}_i \end{cases}$$

$$X_i := \{x_i \in \mathbb{R}^{n_i} \mid g_{i,j}(x_i) = 0, \quad g_{i,j}(x_i) \geq 0\}$$

the feasible set of F_i

regular NEP

the feasible set X_i doesn't depend on x_{-i}.

generalized NEP (GNEP)

the feasible set X_i depends on x_{-i}.

GNEP of polynomials (GNEPP)

all f_i and $g_{i,j}$ are polynomials.

Nash Equilibrium Problem

NEP as an optimization problem.

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) = 0 \quad \text{if } j \in \mathcal{E}_i \\ & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \geq 0 \quad \text{if } j \in \mathcal{I}_i \end{cases}$$

$$X_i := \{x_i \in \mathbb{R}^{n_i} \mid g_{i,j}(x_i) = 0, \quad g_{i,j}(x_i) \geq 0\}$$

the feasible set of F_i

regular NEP

the feasible set X_i doesn't depend on x_{-i}.

generalized NEP (GNEP)

the feasible set X_i depends on x_{-i}.

GNEP of polynomials (GNEPP)

all f_i and $g_{i,j}$ are polynomials.

Nash Equilibrium Problem

NEP as an optimization problem.

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) = 0 \quad \text{if } j \in \mathscr{E}_i \\ & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \geq 0 \quad \text{if } j \in \mathscr{I}_i \end{cases}$$

$$X_i := \{x_i \in \mathbb{R}^{n_i} \mid g_{i,j}(x_i) = 0, \quad g_{i,j}(x_i) \geq 0\}$$

the feasible set of F_i

regular NEP

the feasible set X_i doesn't depend on x_{-i}.

generalized NEP (GNEP)

the feasible set X_i depends on x_{-i}.

GNEP of polynomials (GNEPP)

all f_i and $g_{i,j}$ are polynomials.

Nash Equilibrium Problem

NEP as an optimization problem.

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_N) = 0 \quad \text{if } j \in \mathcal{E}_i \\ & g_{i,j}(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_N) \geq 0 \quad \text{if } j \in \mathcal{I}_i \end{cases}$$

$$X_i := \{x_i \in \mathbb{R}^{n_i} \mid g_{i,j}(x_i) = 0, \quad g_{i,j}(x_i) \geq 0\}$$

the feasible set of F_i

regular NEP

the feasible set X_i doesn't depend on x_{-i}.

generalized NEP (GNEP)

the feasible set X_i depends on x_{-i}.

GNEP of polynomials (GNEPP)

all f_i and $g_{i,j}$ are polynomials.

Nash Equilibrium Problem

NEP as an optimization problem.

$$F_i : \begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \\ \text{s.t.} & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) = 0 \quad \text{if } j \in \mathcal{E}_i \\ & g_{i,j}(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_N) \geq 0 \quad \text{if } j \in \mathcal{I}_i \end{cases}$$

$$X_i := \{x_i \in \mathbb{R}^{n_i} \mid g_{i,j}(x_i) = 0, \quad g_{i,j}(x_i) \geq 0\}$$

the feasible set of F_i

regular NEP

the feasible set X_i doesn't depend on x_{-i}.

generalized NEP (GNEP)

the feasible set X_i depends on x_{-i}.

GNEP of polynomials (GNEPP)

all f_i and $g_{i,j}$ are polynomials.

Optimality Conditions for GNEPs

KKT system

If x_i is a minimizer of F_i, then there is a Lagrange multiplier vector $\lambda_i := (\lambda_{i,1}, \ldots, \lambda_{i,m_i})$ satisfying the first-order **Karush-Kuhn-Tucker (KKT) condition**.

$$\begin{cases} \nabla_{x_i} f_i(x) - \sum_{j=1}^{m_i} \lambda_{i,j} \nabla_{x_i} g_{i,j}(x) = 0 \\ \lambda_{i,j} g_{i,j}(x) = 0 \quad \text{for all } j \\ g_{i,j}(x) = 0 \quad \text{if } j \in \mathscr{E}_i \\ g_{i,j}(x) \geq 0 \text{ and } \lambda_{i,j} \geq 0 \quad \text{if } j \in \mathscr{I}_i \end{cases}$$

Optimality Conditions for GNEPs

KKT system

If x_i is a minimizer of F_i, then there is a Lagrange multiplier vector $\lambda_i := (\lambda_{i,1}, \dots, \lambda_{i,m_i})$ satisfying the first-order **Karush-Kuhn-Tucker (KKT) condition**.

$$\begin{cases} \nabla_{x_i} f_i(x) - \sum_{j=1}^{m_i} \lambda_{i,j} \nabla_{x_i} g_{i,j}(x) = 0 \\ \lambda_{i,j} g_{i,j}(x) = 0 \quad \text{for all } j \\ g_{i,j}(x) = 0 \quad \text{if } j \in \mathscr{E}_i \\ g_{i,j}(x) \geq 0 \text{ and } \lambda_{i,j} \geq 0 \quad \text{if } j \in \mathscr{I}_i \end{cases}$$

Optimality Conditions for GNEPs

KKT system

If x is a generalized Nash equilibrium, then the KKT condition holds for all $i = 1, \ldots, N$.

Then, we have the following KKT system for each $i = 1, \ldots, N$.

$$F_i := \begin{cases} \nabla_{x_i} f_i(x) - \sum_{j=1}^{m_i} \lambda_{i,j} \nabla_{x_i} g_{i,j}(x) = 0 \\ g_{i,j}(x) = 0 \quad \text{for } j = 1, \ldots, m_i \end{cases}$$

Find all solutions of the KKT system $F := (F_1, \ldots, F_N)$.
(a posteriori NE selection required)

Optimality Conditions for GNEPs

KKT system

If x is a generalized Nash equilibrium, then the KKT condition holds for all $i = 1,\ldots,N$.

Then, we have the following KKT system for each $i = 1,\ldots,N$.

$$F_i := \begin{cases} \nabla_{x_i} f_i(x) - \sum_{j=1}^{m_i} \lambda_{i,j} \nabla_{x_i} g_{i,j}(x) = 0 \\ g_{i,j}(x) = 0 \quad \text{for } j = 1,\ldots,m_i \end{cases}$$

Find all solutions of the KKT system $F := (F_1, \ldots, F_N)$.

(a posteriori NE selection required)

Optimality Conditions for GNEPs

KKT system

If x is a generalized Nash equilibrium, then the KKT condition holds for all $i = 1, \ldots, N$.

Then, we have the following KKT system for each $i = 1, \ldots, N$.

$$F_i := \begin{cases} \nabla_{x_i} f_i(x) - \sum_{j=1}^{m_i} \lambda_{i,j} \nabla_{x_i} g_{i,j}(x) = 0 \\ g_{i,j}(x) = 0 \quad \text{for } j = 1, \ldots, m_i \end{cases}$$

Find all solutions of the KKT system $F := (F_1, \ldots, F_N)$.
(a posteriori NE selection required)

Optimality Conditions for GNEPs

Example

Consider 2-player GNEP

$$1\text{st player}: \begin{cases} \min\limits_{x_1 \in \mathbb{R}^1} & \frac{1}{2}x_1^2 x_2^3 - x_1 x_2^2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1 x_2 \geq 0 \end{cases} \qquad 2\text{nd player}: \begin{cases} \min\limits_{x_2 \in \mathbb{R}^1} & \frac{1}{2}x_1^3 x_2^2 - x_1^2 x_2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$$

Optimality Conditions for GNEPs

Example

Consider 2-player GNEP

1st player : $\begin{cases} \min\limits_{x_1 \in \mathbb{R}^1} & \frac{1}{2}x_1^2 x_2^3 - x_1 x_2^2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1 x_2 \geq 0 \end{cases}$ 2nd player : $\begin{cases} \min\limits_{x_2 \in \mathbb{R}^1} & \frac{1}{2}x_1^3 x_2^2 - x_1^2 x_2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$

$$F_1 : \begin{cases} \nabla_{x_1} f_1 - \lambda_1 \nabla_{x_1} g_{1,1} = x_1 x_2^3 - x_2^2 - 2x_2 - \lambda_1(-x_2) \\ \end{cases}$$

Optimality Conditions for GNEPs

Example

Consider 2-player GNEP

1st player : $\begin{cases} \min\limits_{x_1 \in \mathbb{R}^1} & \frac{1}{2}x_1^2x_2^3 - x_1x_2^2 - 2x_1x_2 \\ \text{s.t.} & 1 - x_1x_2 \geq 0 \end{cases}$

2nd player : $\begin{cases} \min\limits_{x_2 \in \mathbb{R}^1} & \frac{1}{2}x_1^3x_2^2 - x_1^2x_2 - 2x_1x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$

$$F_1 : \begin{cases} \nabla_{x_1} f_1 - \lambda_1 \nabla_{x_1} g_{1,1} = x_1x_2^3 - x_2^2 - 2x_2 - \lambda_1(-x_2) \\ \lambda_1 g_{1,1} = \lambda_1(1 - x_1x_2) \end{cases}$$

Optimality Conditions for GNEPs

Example

Consider 2-player GNEP

1st player : $\begin{cases} \min\limits_{x_1 \in \mathbb{R}^1} & \frac{1}{2}x_1^2 x_2^3 - x_1 x_2^2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1 x_2 \geq 0 \end{cases}$ 2nd player : $\begin{cases} \min\limits_{x_2 \in \mathbb{R}^1} & \frac{1}{2}x_1^3 x_2^2 - x_1^2 x_2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$

$$F_1 : \begin{cases} \nabla_{x_1} f_1 - \lambda_1 \nabla_{x_1} g_{1,1} = x_1 x_2^3 - x_2^2 - 2x_2 - \lambda_1(-x_2) \\ \lambda_1 g_{1,1} = \lambda_1(1 - x_1 x_2) \end{cases}$$

Optimality Conditions for GNEPs

Example

Consider 2-player GNEP

$$1\text{st player}: \begin{cases} \min\limits_{x_1 \in \mathbb{R}^1} & \frac{1}{2}x_1^2 x_2^3 - x_1 x_2^2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1 x_2 \geq 0 \end{cases} \qquad 2\text{nd player}: \begin{cases} \min\limits_{x_2 \in \mathbb{R}^1} & \frac{1}{2}x_1^3 x_2^2 - x_1^2 x_2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$$

$$F_1 : \begin{cases} \nabla_{x_1} f_1 - \lambda_1 \nabla_{x_1} g_{1,1} = x_1 x_2^3 - x_2^2 - 2x_2 - \lambda_1(-x_2) \\ \lambda_1 g_{1,1} = \lambda_1(1 - x_1 x_2) \end{cases} \qquad F_2 : \begin{cases} \nabla_{x_2} f_2 - \lambda_2 \nabla_{x_2} g_{2,1} = x_1^3 x_2 - x_1^2 - 2x_1 - \lambda_2(-2x_2) \end{cases}$$

Optimality Conditions for GNEPs

Example

Consider 2-player GNEP

$$\text{1st player}: \begin{cases} \min\limits_{x_1 \in \mathbb{R}^1} & \frac{1}{2}x_1^2 x_2^3 - x_1 x_2^2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1 x_2 \geq 0 \end{cases} \qquad \text{2nd player}: \begin{cases} \min\limits_{x_2 \in \mathbb{R}^1} & \frac{1}{2}x_1^3 x_2^2 - x_1^2 x_2 - 2x_1 x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$$

$$F_1 : \begin{cases} \nabla_{x_1} f_1 - \lambda_1 \nabla_{x_1} g_{1,1} = x_1 x_2^3 - x_2^2 - 2x_2 - \lambda_1(-x_2) \\ \lambda_1 g_{1,1} = \lambda_1(1 - x_1 x_2) \end{cases} \qquad F_2 : \begin{cases} \nabla_{x_2} f_2 - \lambda_2 \nabla_{x_2} g_{2,1} = x_1^3 x_2 - x_1^2 - 2x_1 - \lambda_2(-2x_2) \\ g_{2,1} = 1 - x_1^2 - x_2^2 \end{cases}$$

Optimality Conditions for GNEPs

Example

Consider 2-player GNEP

1st player : $\begin{cases} \min\limits_{x_1 \in \mathbb{R}^1} & \frac{1}{2}x_1^2x_2^3 - x_1x_2^2 - 2x_1x_2 \\ \text{s.t.} & 1 - x_1x_2 \geq 0 \end{cases}$ 2nd player : $\begin{cases} \min\limits_{x_2 \in \mathbb{R}^1} & \frac{1}{2}x_1^3x_2^2 - x_1^2x_2 - 2x_1x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$

$F_1 : \begin{cases} \nabla_{x_1} f_1 - \lambda_1 \nabla_{x_1} g_{1,1} = x_1x_2^3 - x_2^2 - 2x_2 - \lambda_1(-x_2) \\ \lambda_1 g_{1,1} = \lambda_1(1 - x_1x_2) \end{cases}$ $F_2 : \begin{cases} \nabla_{x_2} f_2 - \lambda_2 \nabla_{x_2} g_{2,1} = x_1^3x_2 - x_1^2 - 2x_1 - \lambda_2(-2x_2) \\ g_{2,1} = 1 - x_1^2 - x_2^2 \end{cases}$

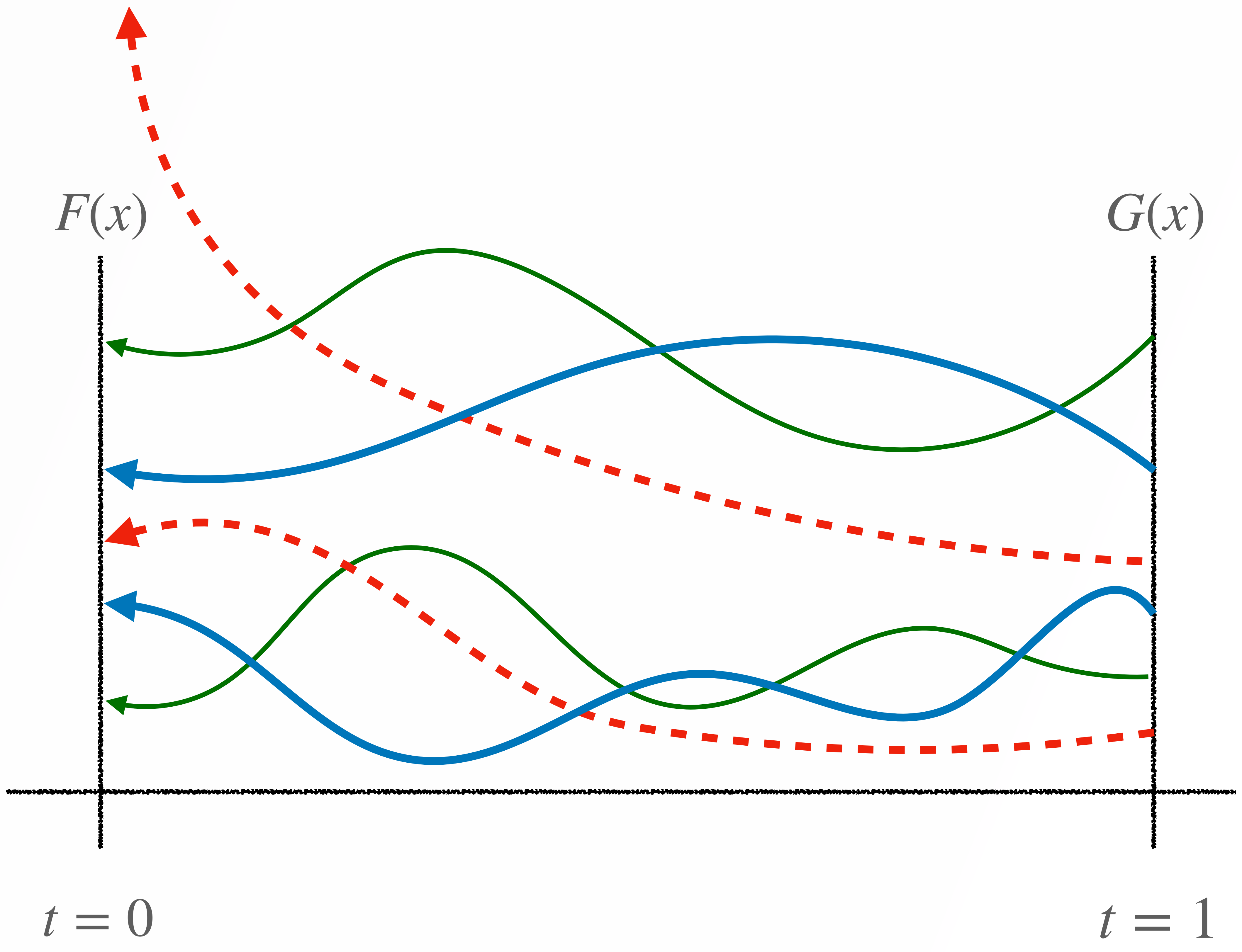
These systems provide the system $F := \{F_1, F_2\}$

Homotopy Continuation

Finding solutions by tracking homotopy

$$H(t, x) = t\gamma G(x) + (1-t)F(x), \quad t \in [0,1]$$

Solve F (**target system**) by constructing a homotopy with G (**start system**) whose solutions are known.

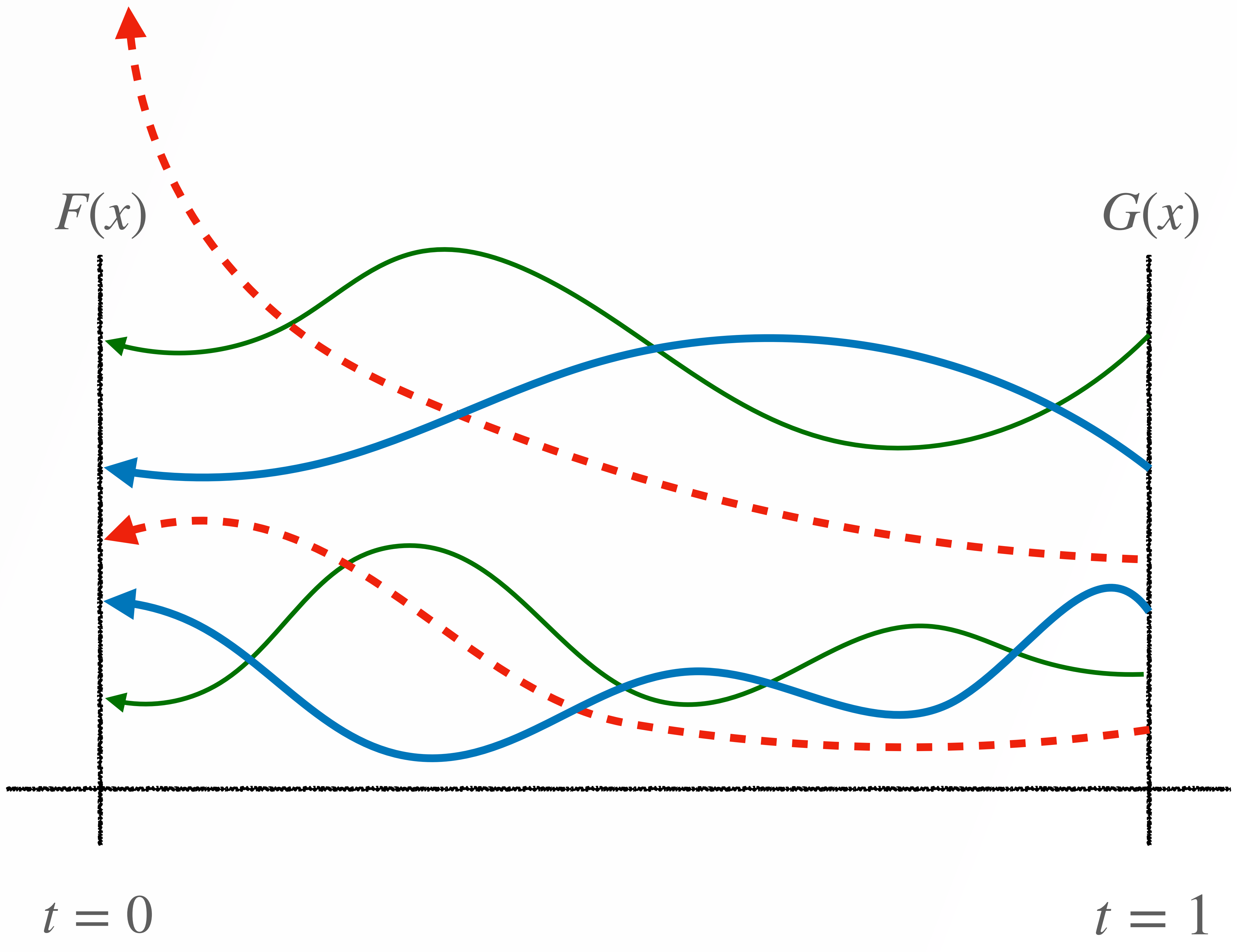


Homotopy Continuation

How to choose start system?

The choice of start system determines the number of homotopy paths to track.

- Bézout homotopy (Bézout bound = product of degrees)
- polyhedral homotopy (BKK bound)
- multihomogeneous start system

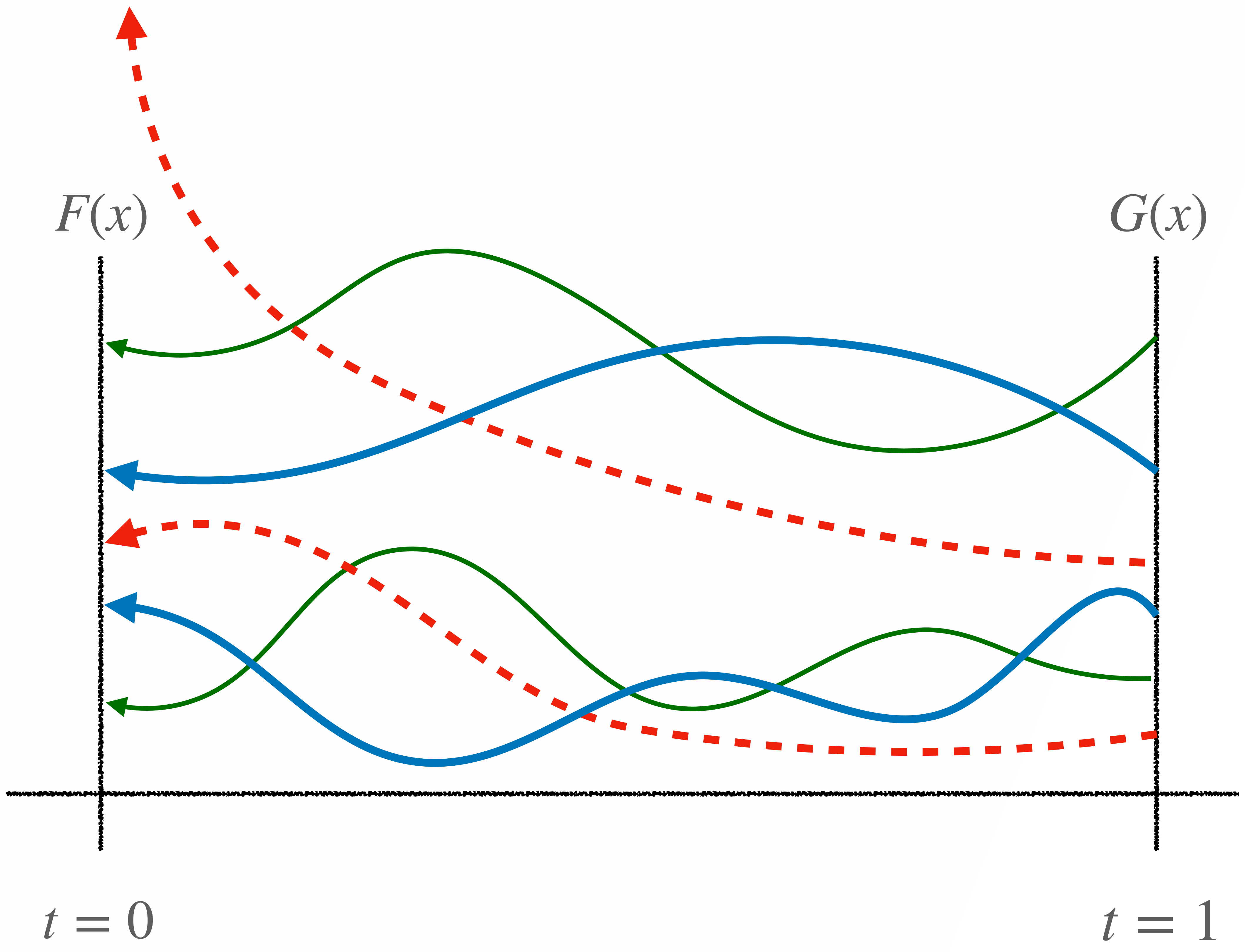


Homotopy Continuation

How to choose start system?

The choice of start system determines the number of homotopy paths to track.

- Bézout homotopy (Bézout bound = product of degrees)
- polyhedral homotopy (BKK bound)
- multihomogeneous start system

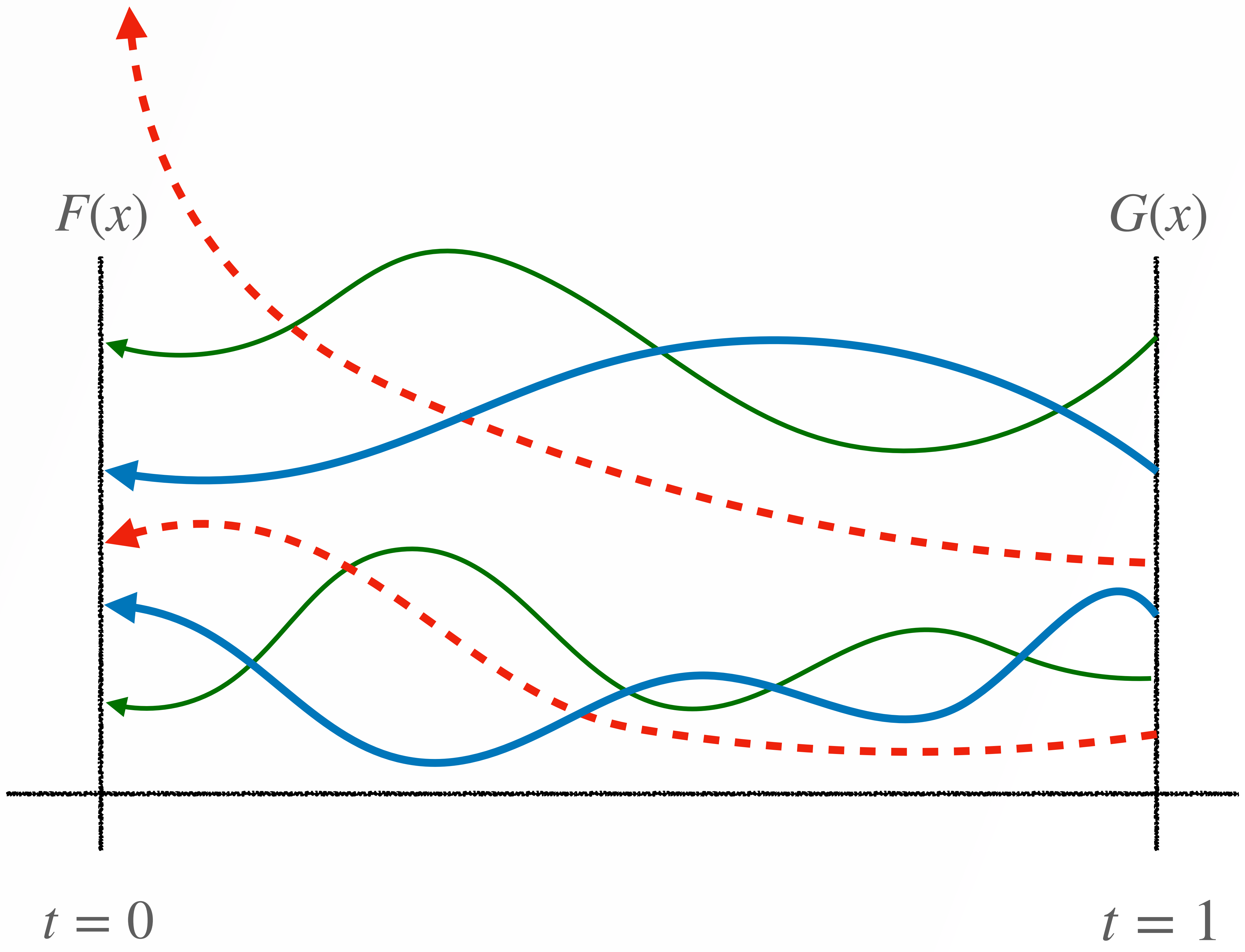


Homotopy Continuation

How to choose start system?

The choice of start system determines the number of homotopy paths to track.

- Bézout homotopy (Bézout bound = product of degrees)
- polyhedral homotopy (BKK bound)
- multihomogeneous start system



Polyhedral Homotopy Continuation

Bernstein theorem

Theorem [Bernstein 1975]. $F := \{f_1, \ldots, f_n\} \subset \mathbb{C}[x_1, \ldots, x_n]$: a square polynomial system.

Q_i : the Newton polytope of f_i. Then, (# isolated roots in $(\mathbb{C} \backslash \{0\})^n) \leq MV(Q_1, \ldots, Q_n)$.

$MV(Q_1, \ldots, Q_n)$ is the **mixed volume** of $Q_1, \ldots, Q_n$.

The mixed volume above is called the **BKK bound**.

The polyhedral homotopy tracks BKK bound many paths.

Polyhedral Homotopy Continuation

Bernstein theorem

Theorem [Bernstein 1975]. $F := \{f_1, \ldots, f_n\} \subset \mathbb{C}[x_1, \ldots, x_n]$: a square polynomial system.

Q_i : the Newton polytope of f_i. Then, $(\#$ isolated roots in $(\mathbb{C}\backslash\{0\})^n) \leq MV(Q_1, \ldots, Q_n)$.

$MV(Q_1, \ldots, Q_n)$ is the **mixed volume** of $Q_1, \ldots, Q_n$.

The mixed volume above is called the **BKK bound**.

The polyhedral homotopy tracks BKK bound many paths.

Polyhedral Homotopy Continuation

Bernstein theorem

Theorem [Bernstein 1975]. $F := \{f_1, \ldots, f_n\} \subset \mathbb{C}[x_1, \ldots, x_n]$: a square polynomial system.

Q_i : the Newton polytope of f_i. Then, $(\#$ isolated roots in $(\mathbb{C}\setminus\{0\})^n) \leq MV(Q_1, \ldots, Q_n)$.

$MV(Q_1, \ldots, Q_n)$ is the **mixed volume** of $Q_1, \ldots, Q_n$.

The mixed volume above is called the **BKK bound**.

The polyhedral homotopy tracks BKK bound many paths.

Polyhedral Homotopy Continuation

Bernstein theorem

Theorem [Bernstein 1975]. $F := \{f_1, \ldots, f_n\} \subset \mathbb{C}[x_1, \ldots, x_n]$: a square polynomial system.

Q_i : the Newton polytope of f_i. Then, $(\#$ isolated roots in $(\mathbb{C}\backslash\{0\})^n) \leq MV(Q_1, \ldots, Q_n)$.

$MV(Q_1, \ldots, Q_n)$ is the **mixed volume** of $Q_1, \ldots, Q_n$.

The mixed volume above is called the **BKK bound**.

The polyhedral homotopy tracks BKK bound many paths.

Polyhedral Homotopy Continuation

Mixed volume

$f \in \mathbb{C}[x_1, \ldots, x_n]$: a polynomial.

$A := \mathrm{supp}(f)$: the **support** of f (the set of exponents of monomials appear in f).

$Q := \mathrm{conv}(A) \subset \mathbb{R}^n$: the **Newton polytope** (the convex hull of A).

$f_1, \ldots, f_n$: polynomials with Newton polytopes $Q_1, \ldots, Q_n$.

$MV(Q_1, \ldots, Q_n)$: the mixed volume of $Q_1, \ldots, Q_n$.

the coefficient of $\lambda_1 \cdots \lambda_n$ term in a polynomial $\mathrm{Vol}(\lambda_1 Q_1 + \cdots + \lambda_n Q_n)$.

Polyhedral Homotopy Continuation

Mixed volume

$f \in \mathbb{C}[x_1, \ldots, x_n]$: a polynomial.

$A := \operatorname{supp}(f)$: the **support** of f (the set of exponents of monomials appear in f).

$Q := \operatorname{conv}(A) \subset \mathbb{R}^n$: the **Newton polytope** (the convex hull of A).

$f_1, \ldots, f_n$: polynomials with Newton polytopes $Q_1, \ldots, Q_n$.

$MV(Q_1, \ldots, Q_n)$: the mixed volume of $Q_1, \ldots, Q_n$.

the coefficient of $\lambda_1 \cdots \lambda_n$ term in a polynomial $\operatorname{Vol}(\lambda_1 Q_1 + \cdots + \lambda_n Q_n)$.

Polyhedral Homotopy Continuation

Mixed volume

$f \in \mathbb{C}[x_1, \ldots, x_n]$: a polynomial.

$A := \mathrm{supp}(f)$: the **support** of f (the set of exponents of monomials appear in f).

$Q := \mathrm{conv}(A) \subset \mathbb{R}^n$: the **Newton polytope** (the convex hull of A).

$f_1, \ldots, f_n$: polynomials with Newton polytopes $Q_1, \ldots, Q_n$.

$MV(Q_1, \ldots, Q_n)$: the mixed volume of $Q_1, \ldots, Q_n$.

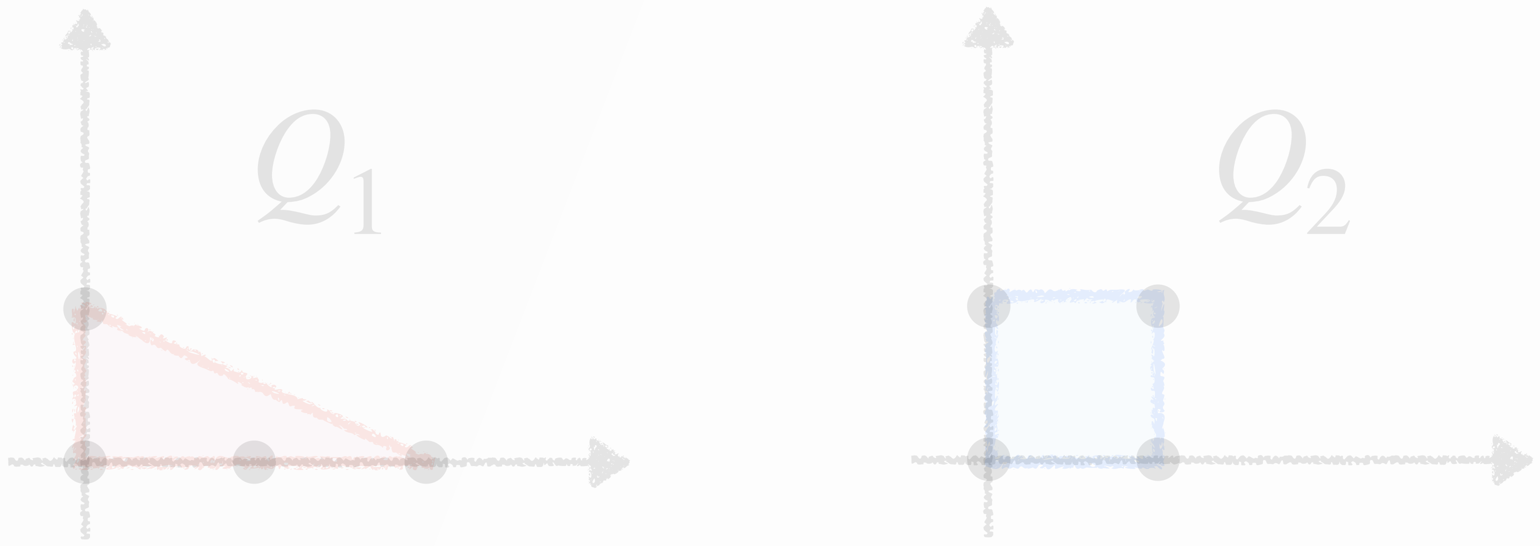
the coefficient of $\lambda_1 \cdots \lambda_n$ term in a polynomial $\mathrm{Vol}(\lambda_1 Q_1 + \cdots + \lambda_n Q_n)$.

Polyhedral Homotopy Continuation

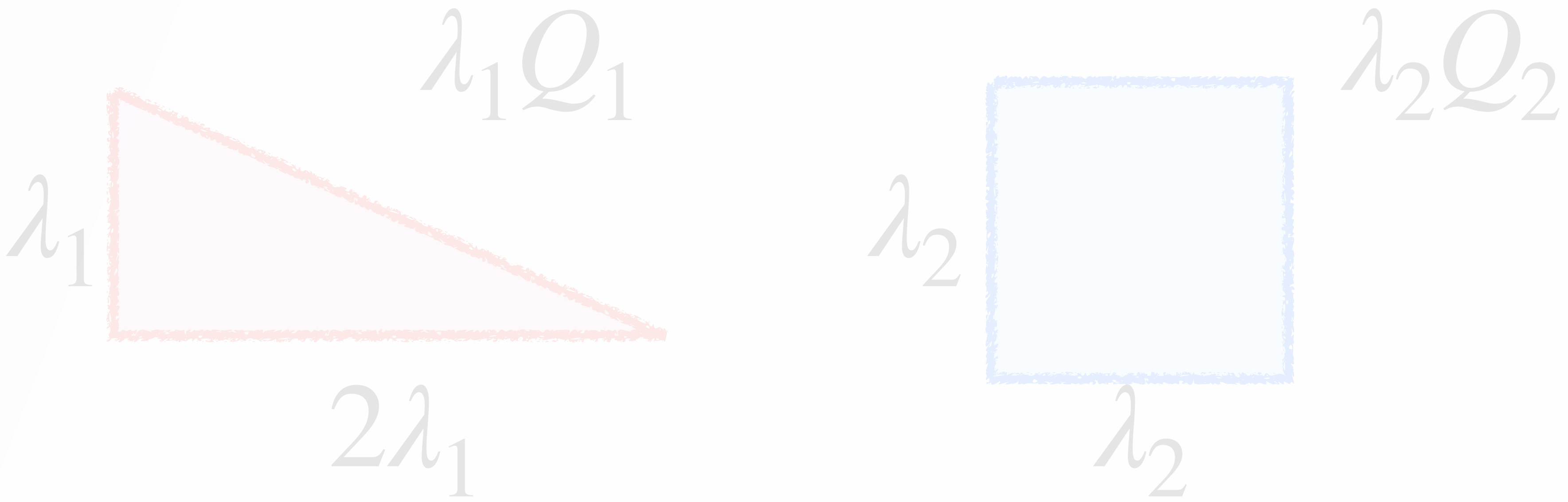
Example $f_1(x, y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x, y) = a_2xy + b_2x + c_2y + d_2$

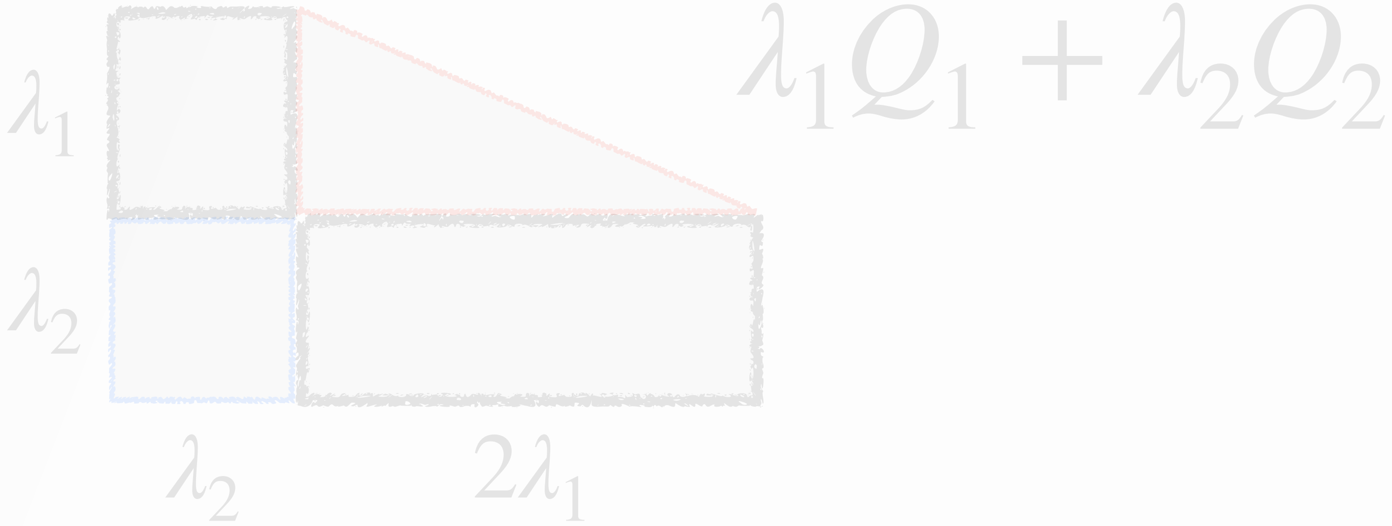
$\Rightarrow A_1 = \{(2,0), (1,0), (0,1), (0,0)\}$ and $A_2 = \{(1,1), (1,0), (0,1), (0,0)\}$

$\Rightarrow$



$\Rightarrow$





$\Rightarrow$ Vol$(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$

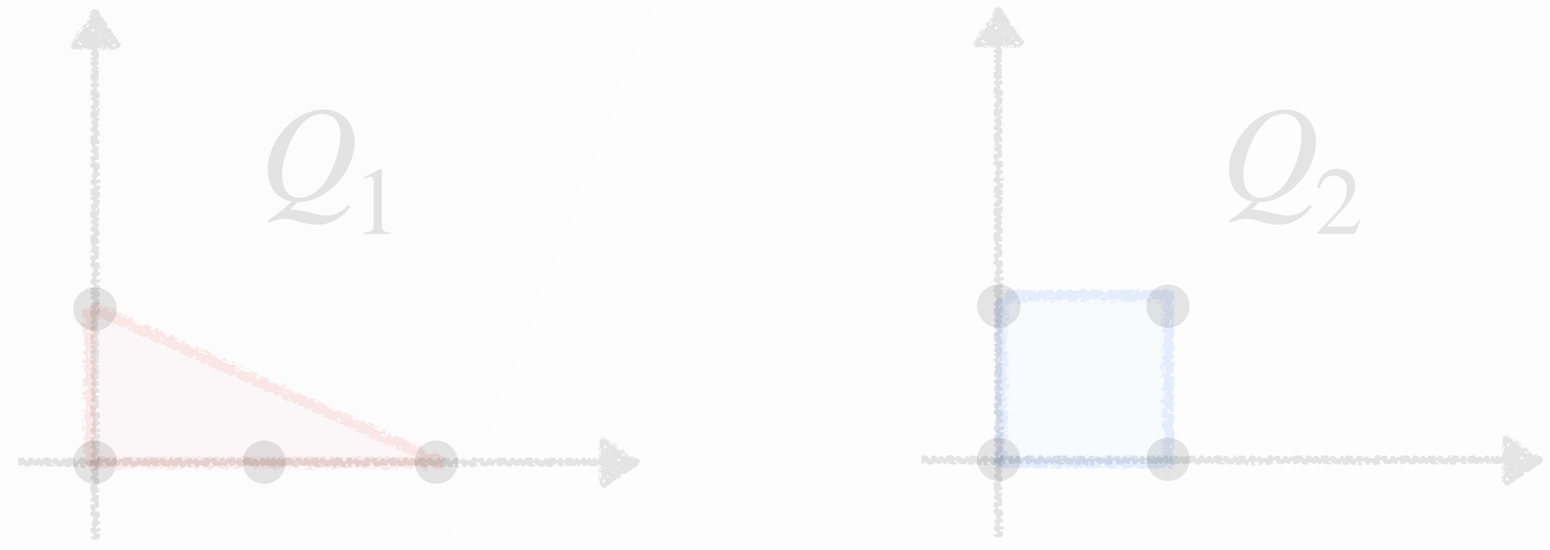
$\Rightarrow MV(Q_1, Q_2) = 3$

Polyhedral Homotopy Continuation

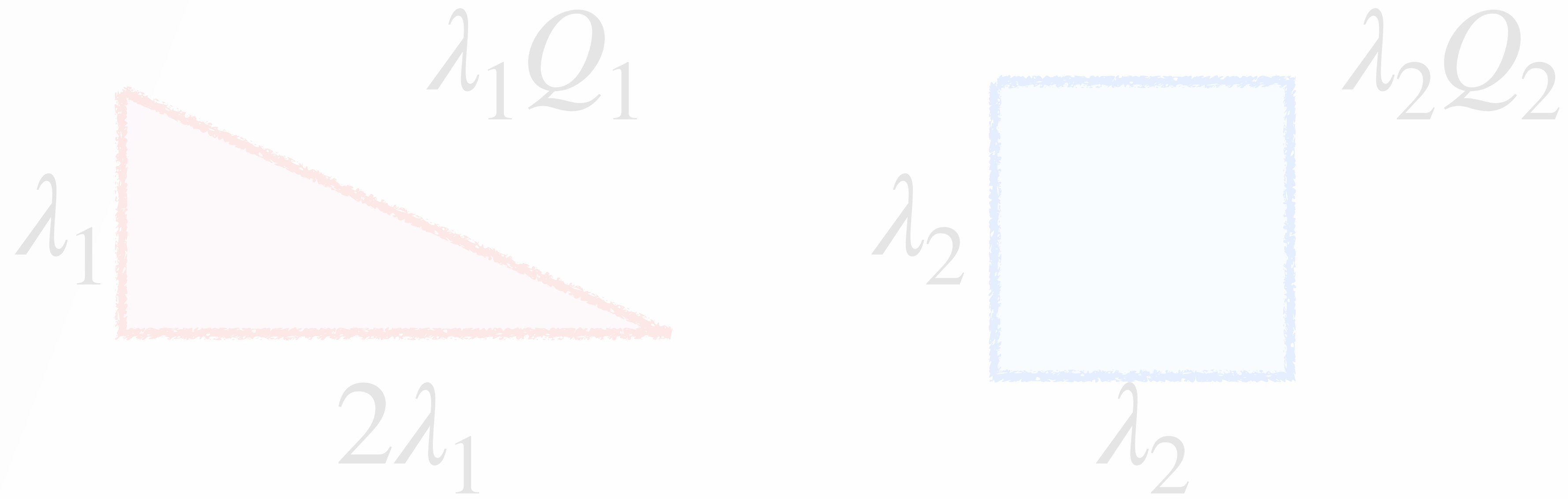
Example $f_1(x, y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x, y) = a_2xy + b_2x + c_2y + d_2$

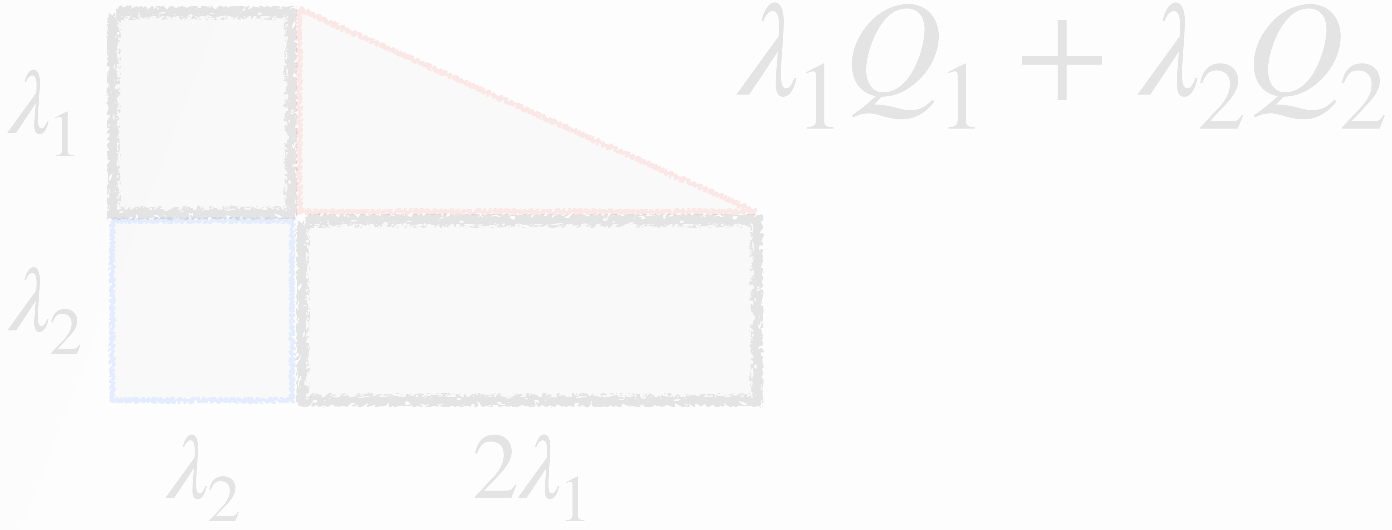
$\Rightarrow A_1 = \{(2,0), (1,0), (0,1), (0,0)\}$ and $A_2 = \{(1,1), (1,0), (0,1), (0,0)\}$

$\Rightarrow$



$\Rightarrow$





$\Rightarrow \text{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$

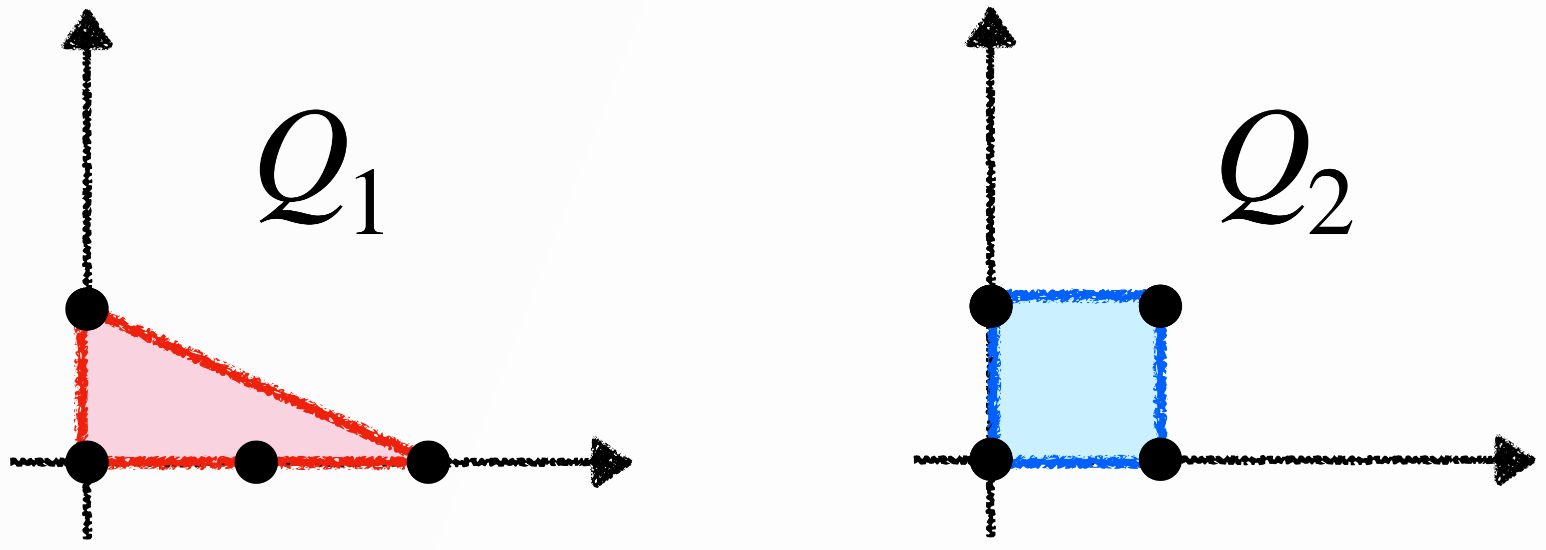
$\Rightarrow MV(Q_1, Q_2) = 3$

Polyhedral Homotopy Continuation

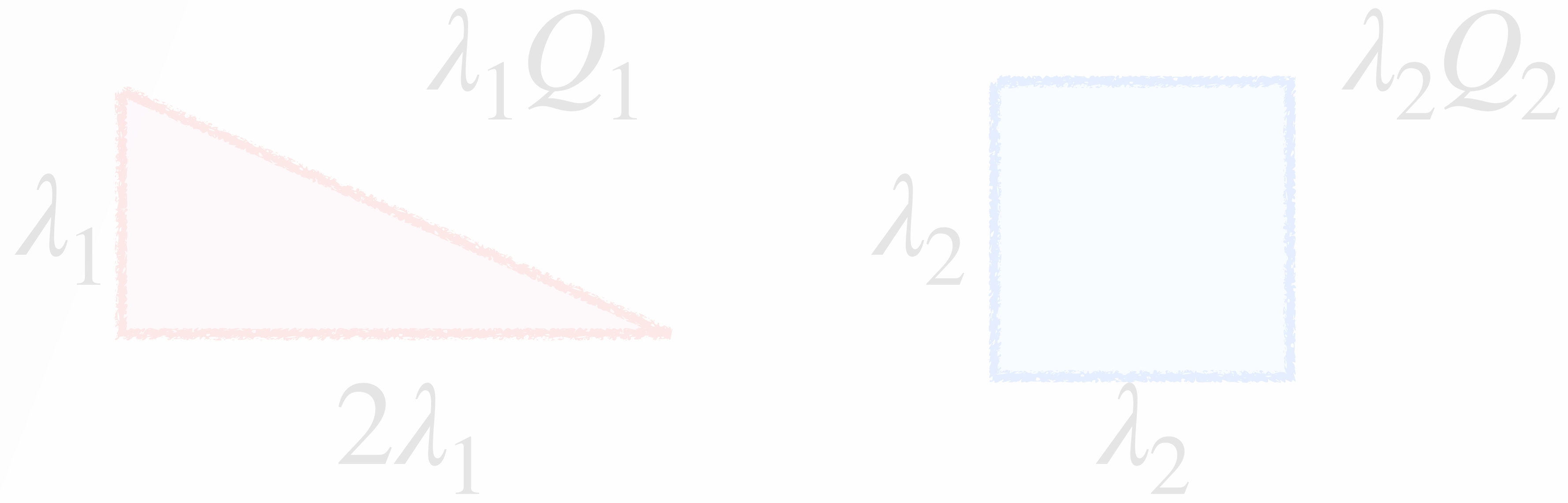
Example $f_1(x, y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x, y) = a_2xy + b_2x + c_2y + d_2$

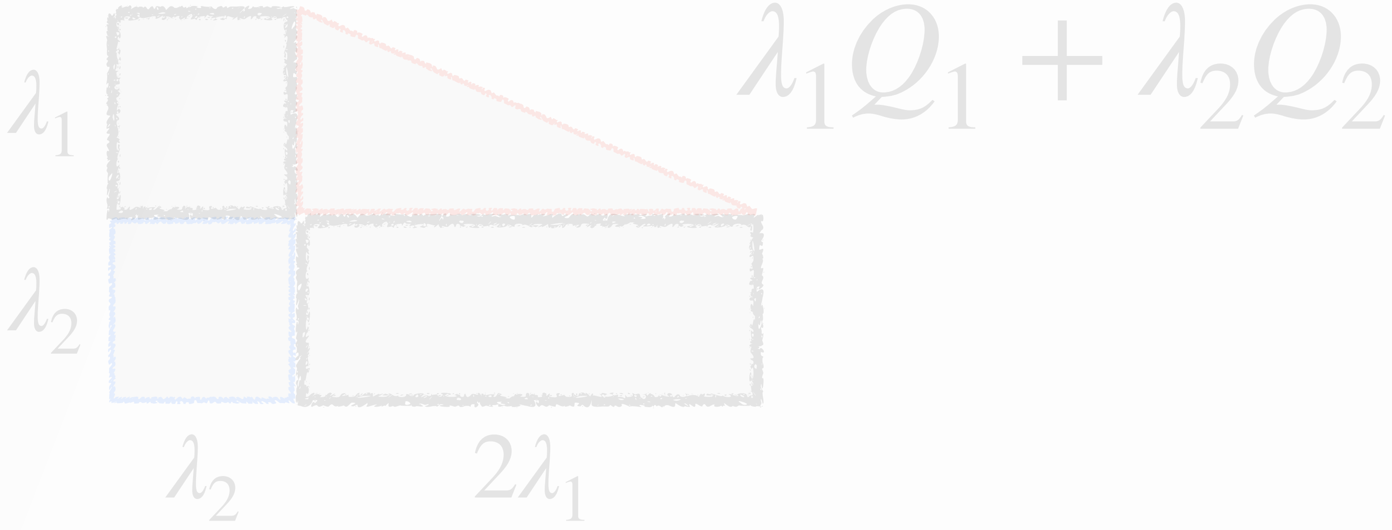
$\Rightarrow A_1 = \{(2,0), (1,0), (0,1), (0,0)\}$ and $A_2 = \{(1,1), (1,0), (0,1), (0,0)\}$

$\Rightarrow$



$\Rightarrow$





$\Rightarrow \text{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$

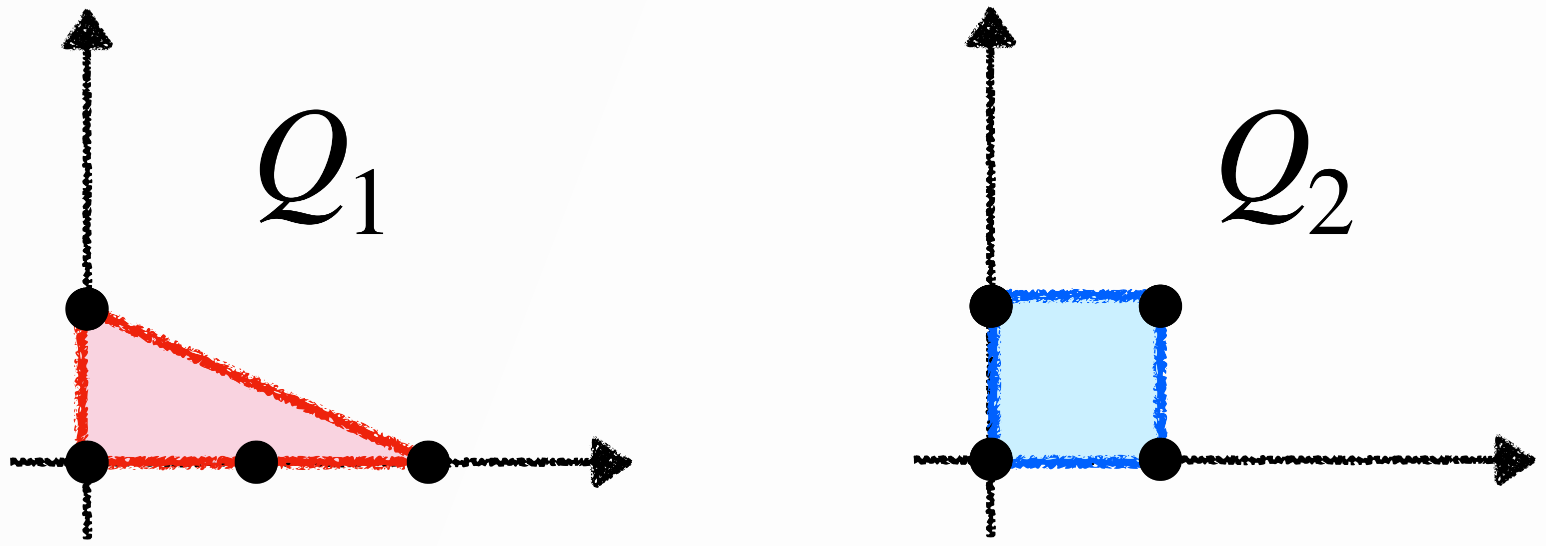
$\Rightarrow MV(Q_1, Q_2) = 3$

Polyhedral Homotopy Continuation

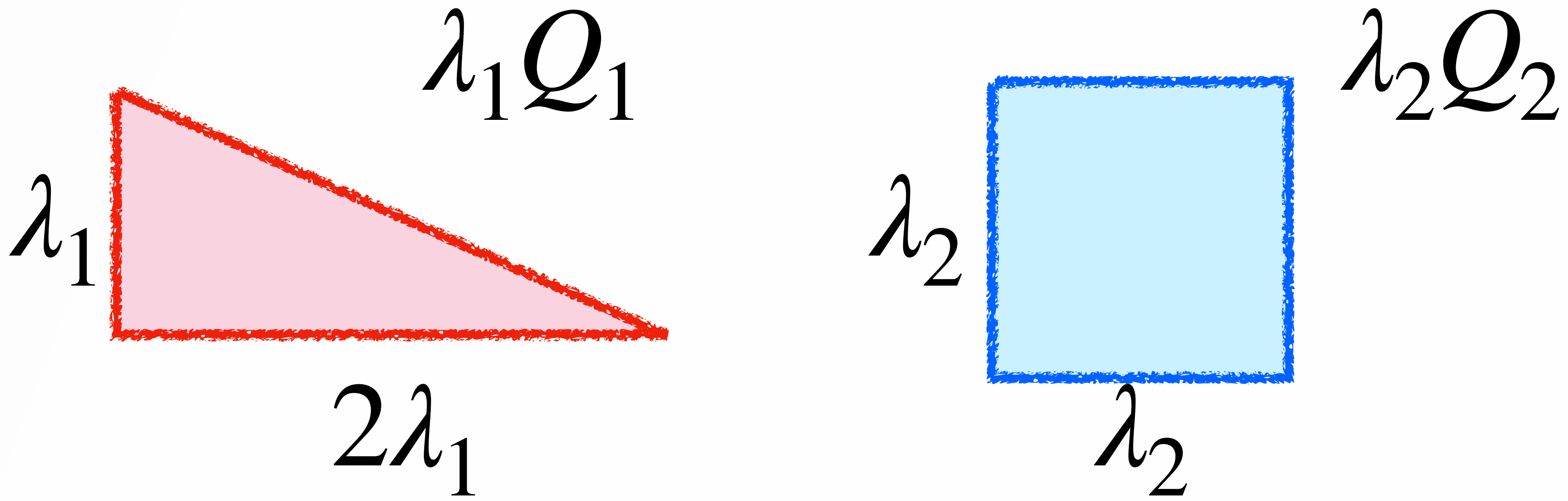
Example $f_1(x, y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x, y) = a_2xy + b_2x + c_2y + d_2$

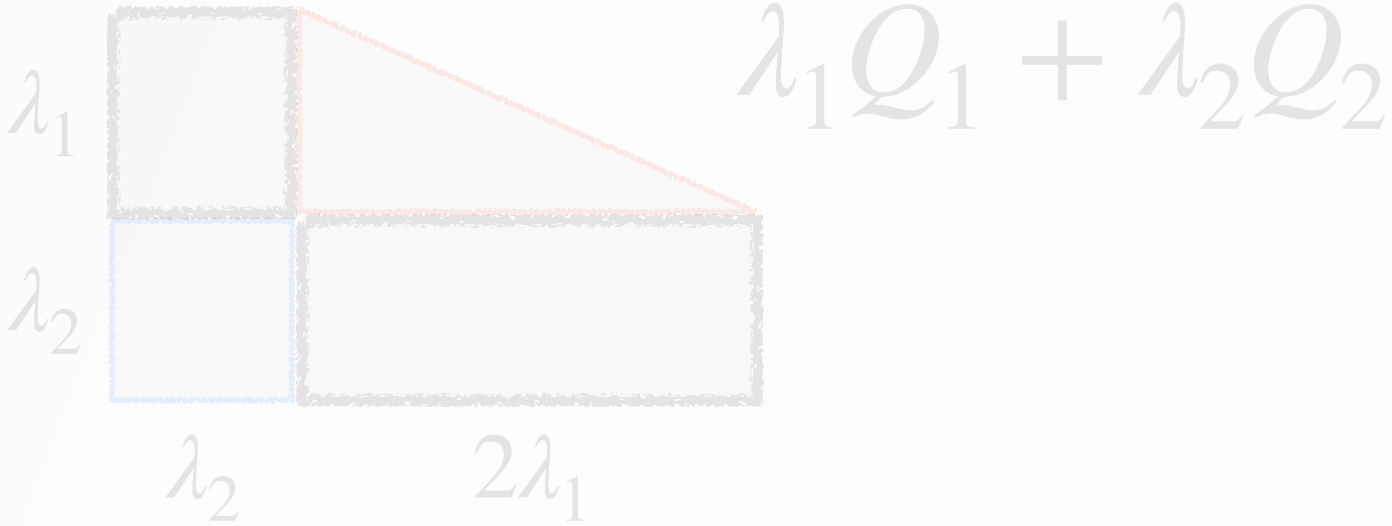
$\Rightarrow A_1 = \{(2,0), (1,0), (0,1), (0,0)\}$ and $A_2 = \{(1,1), (1,0), (0,1), (0,0)\}$

$\Rightarrow$



$\Rightarrow$





$\Rightarrow \text{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$

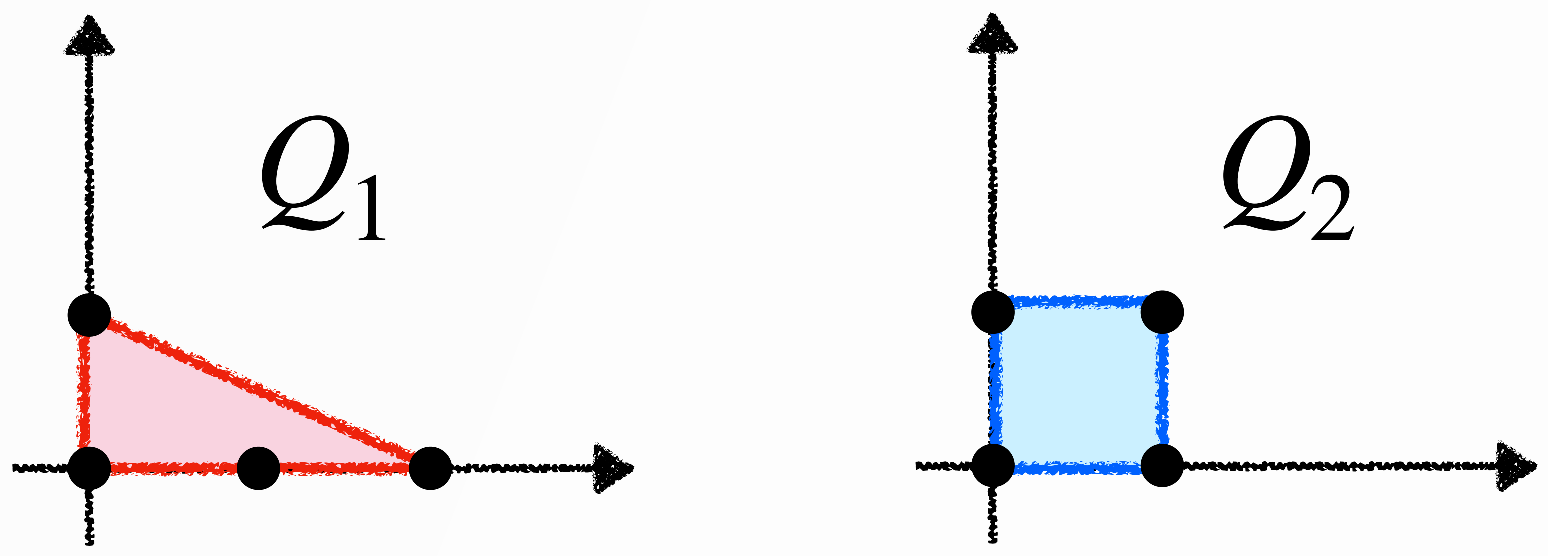
$\Rightarrow MV(Q_1, Q_2) = 3$

Polyhedral Homotopy Continuation

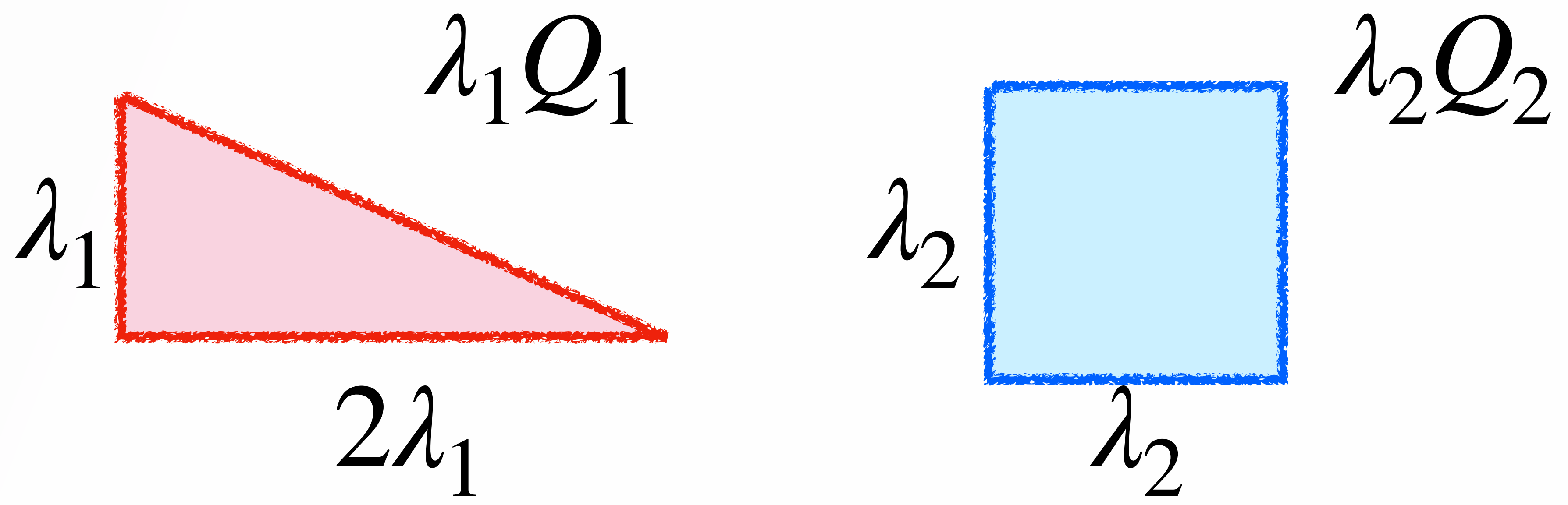
Example $f_1(x,y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x,y) = a_2xy + b_2x + c_2y + d_2$

$\Rightarrow A_1 = \{(2{,}0),(1{,}0),(0{,}1),(0{,}0)\}$ and $A_2 = \{(1{,}1),(1{,}0),(0{,}1),(0{,}0)\}$

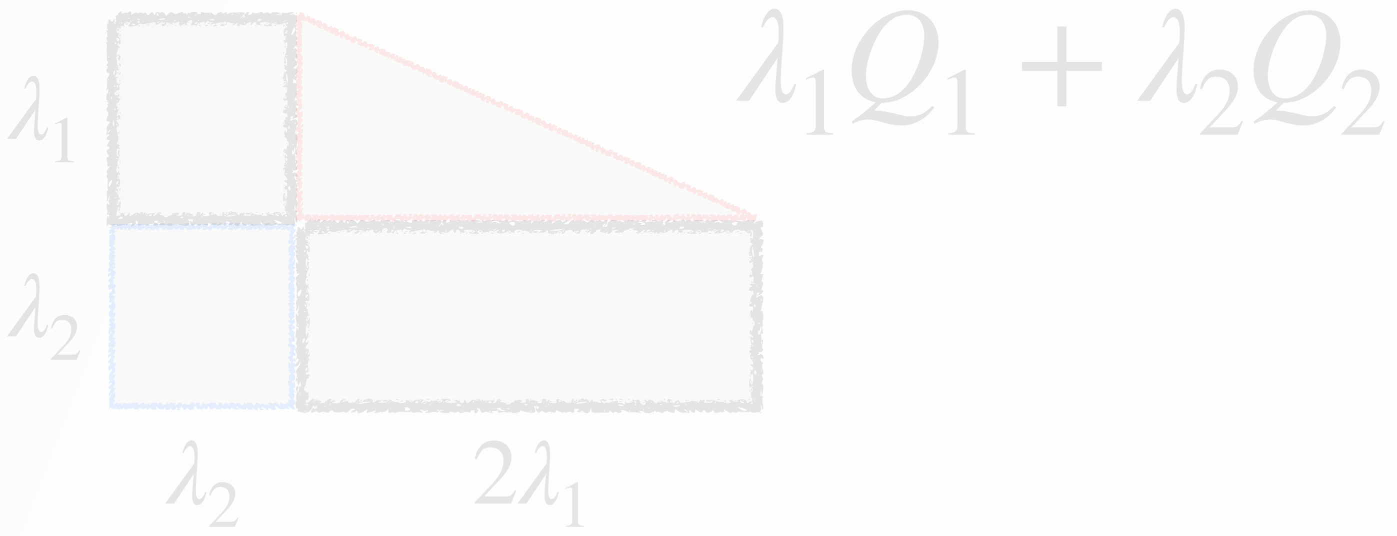
$\Rightarrow$



$\Rightarrow$



$\Rightarrow \text{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$



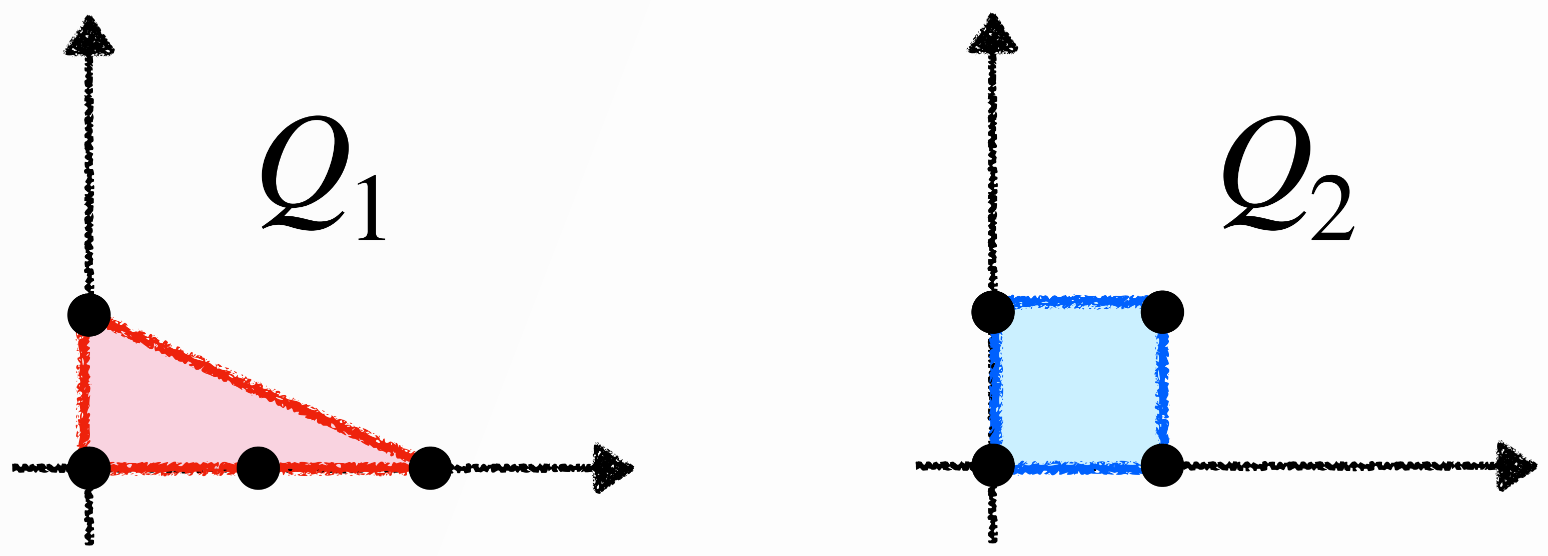
$\Rightarrow MV(Q_1, Q_2) = 3$

Polyhedral Homotopy Continuation

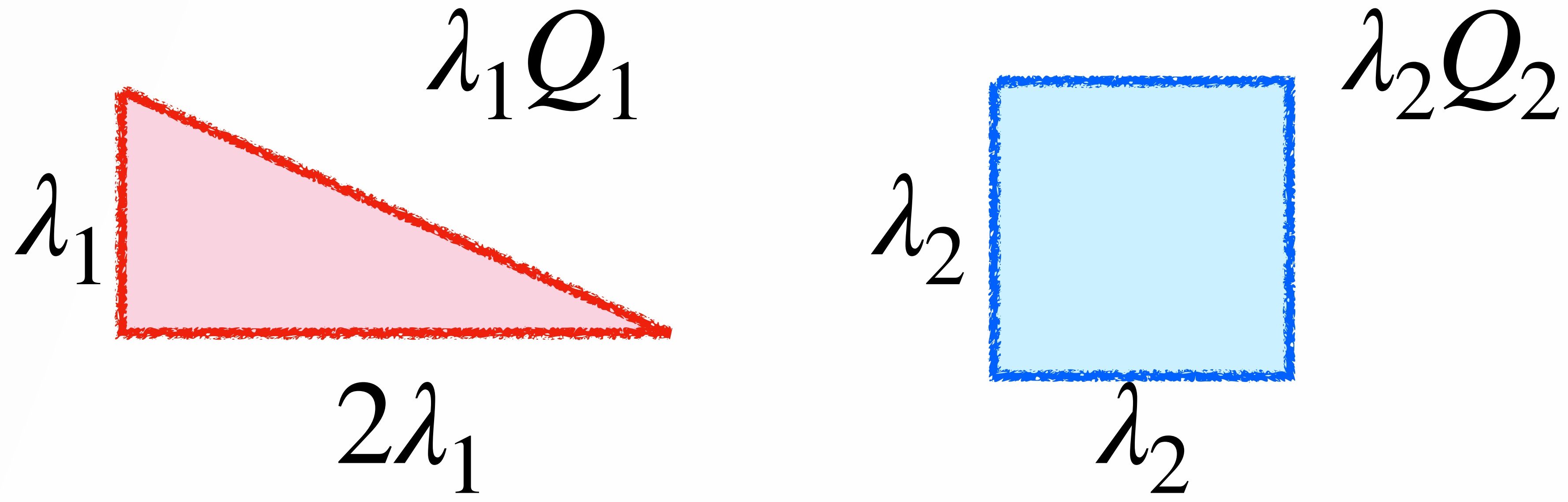
Example $f_1(x,y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x,y) = a_2xy + b_2x + c_2y + d_2$

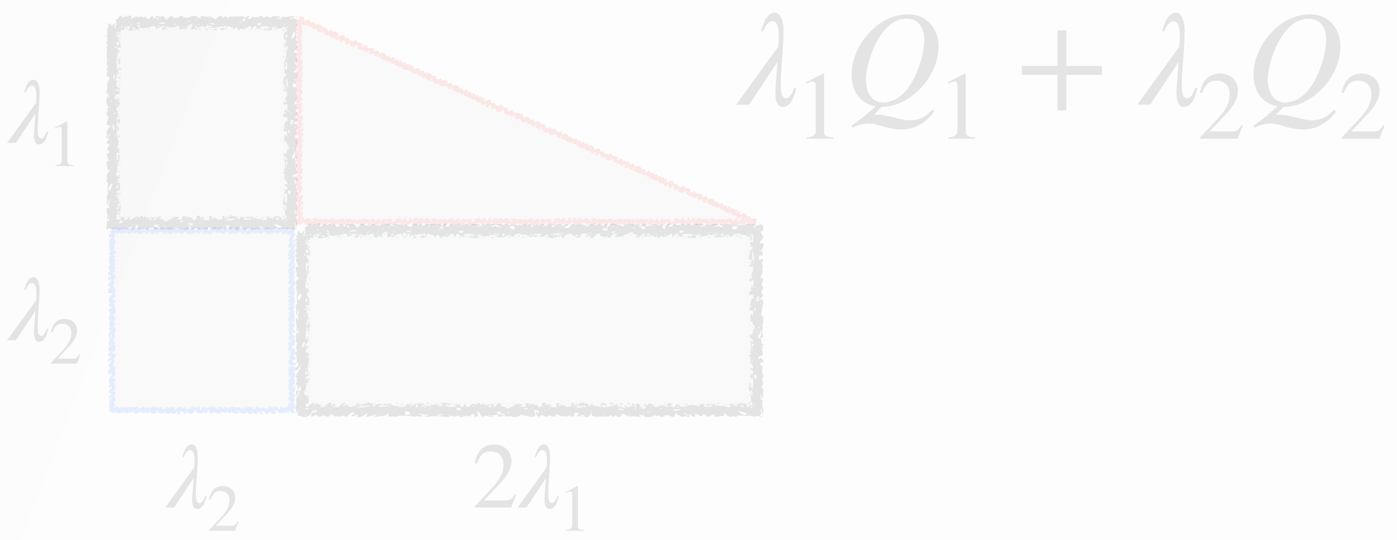
$\Rightarrow A_1 = \{(2{,}0),(1{,}0),(0{,}1),(0{,}0)\}$ and $A_2 = \{(1{,}1),(1{,}0),(0{,}1),(0{,}0)\}$

$\Rightarrow$



$\Rightarrow$





$\Rightarrow \text{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$

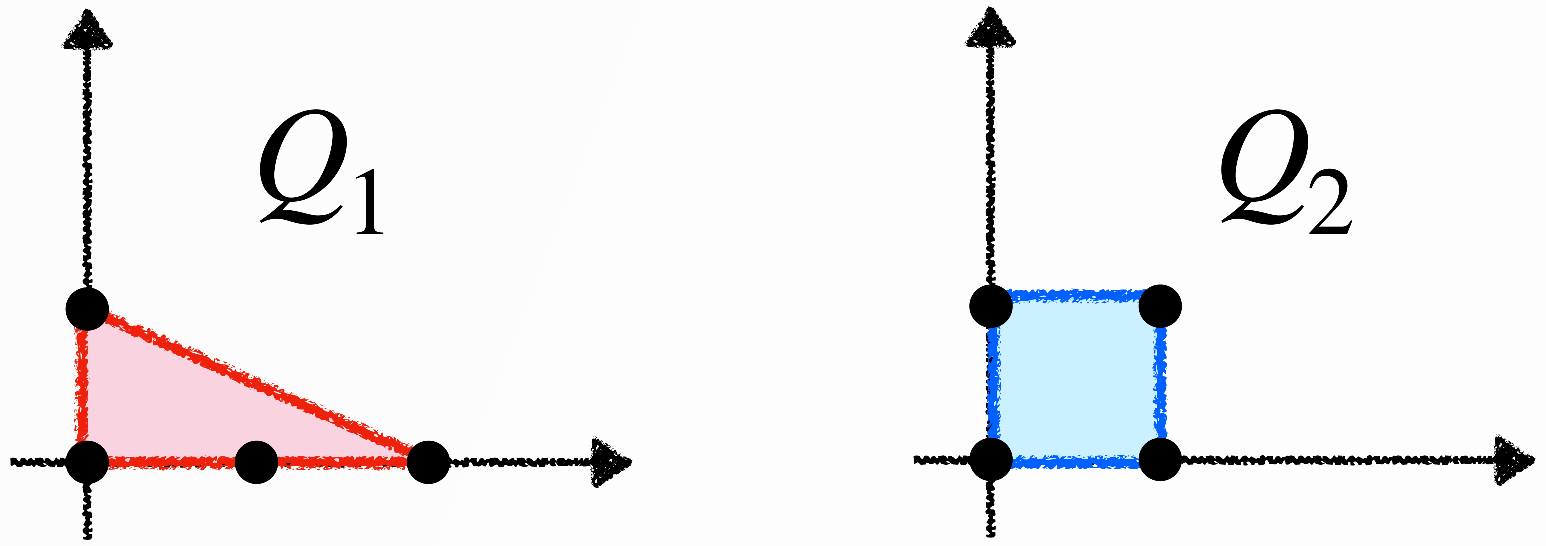
$\Rightarrow MV(Q_1, Q_2) = 3$

Polyhedral Homotopy Continuation

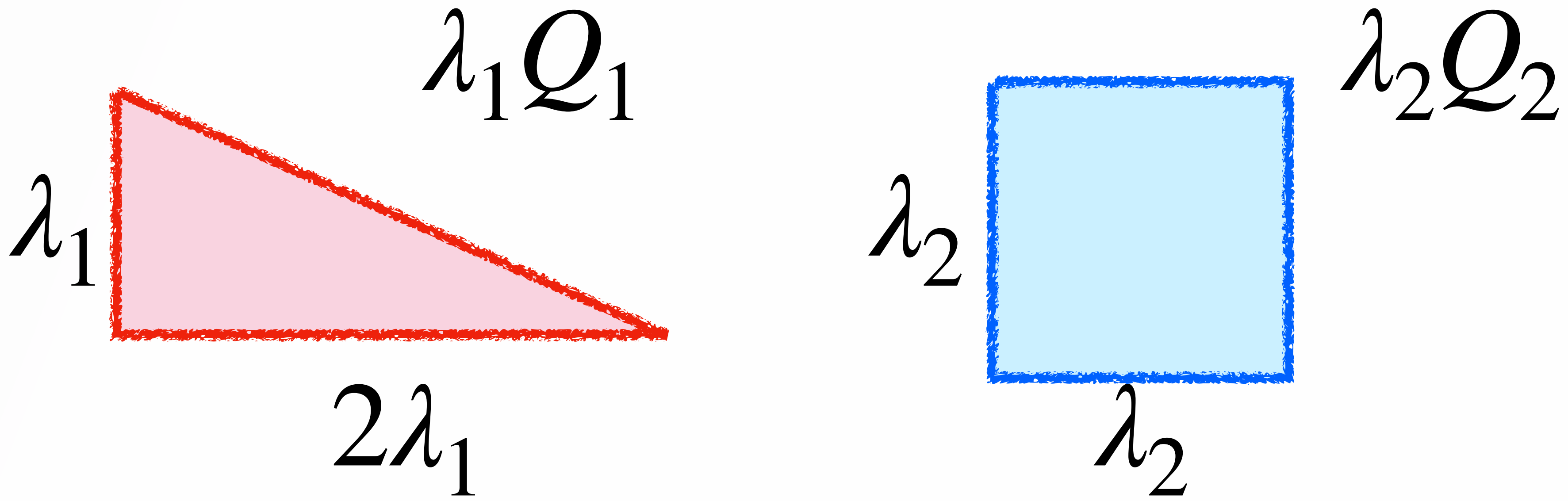
Example $f_1(x, y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x, y) = a_2xy + b_2x + c_2y + d_2$

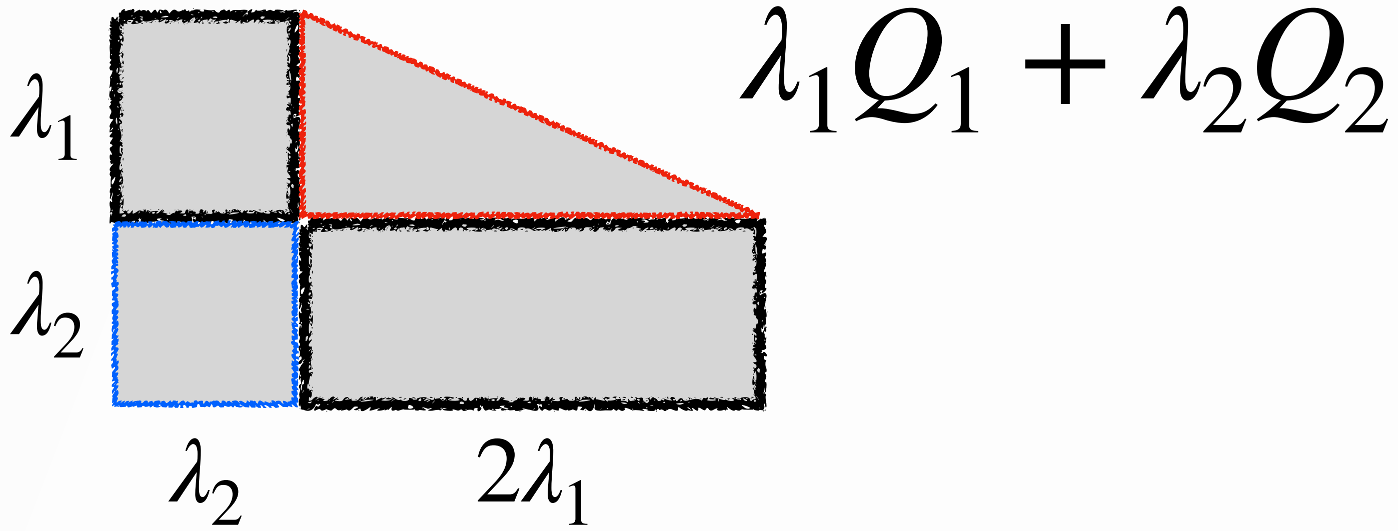
$\Rightarrow A_1 = \{(2,0), (1,0), (0,1), (0,0)\}$ and $A_2 = \{(1,1), (1,0), (0,1), (0,0)\}$

$\Rightarrow$



$\Rightarrow$





$\Rightarrow \mathrm{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$

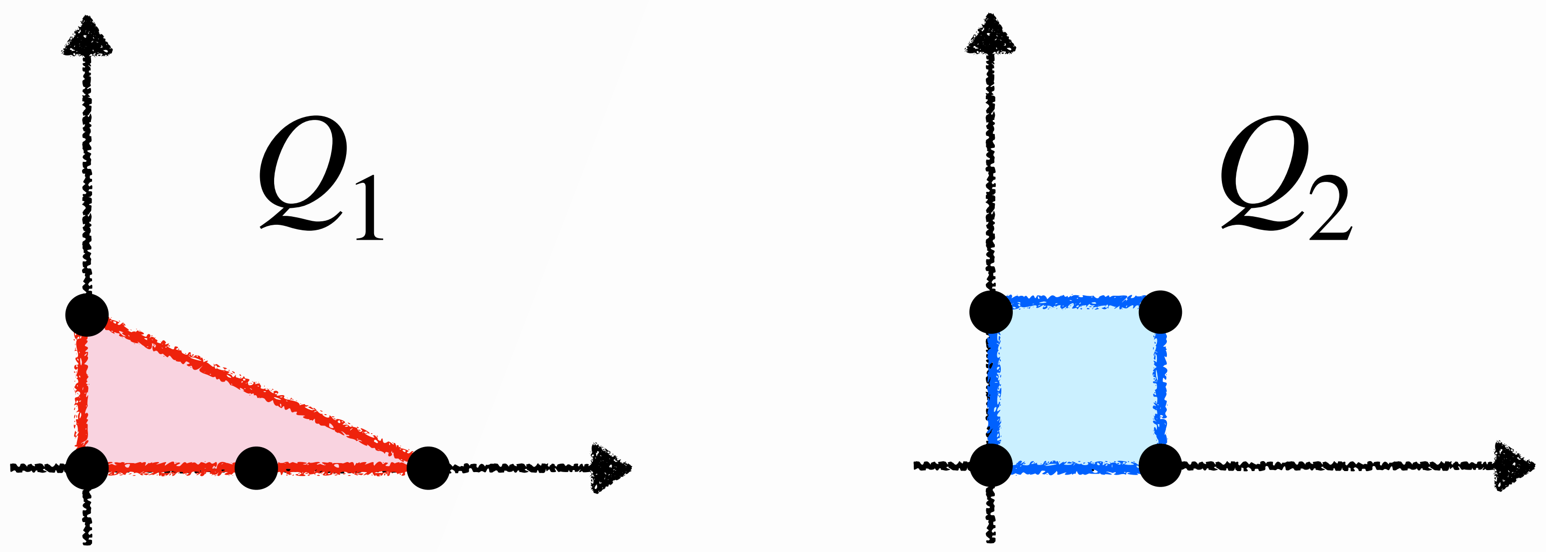
$\Rightarrow MV(Q_1, Q_2) = 3$

Polyhedral Homotopy Continuation

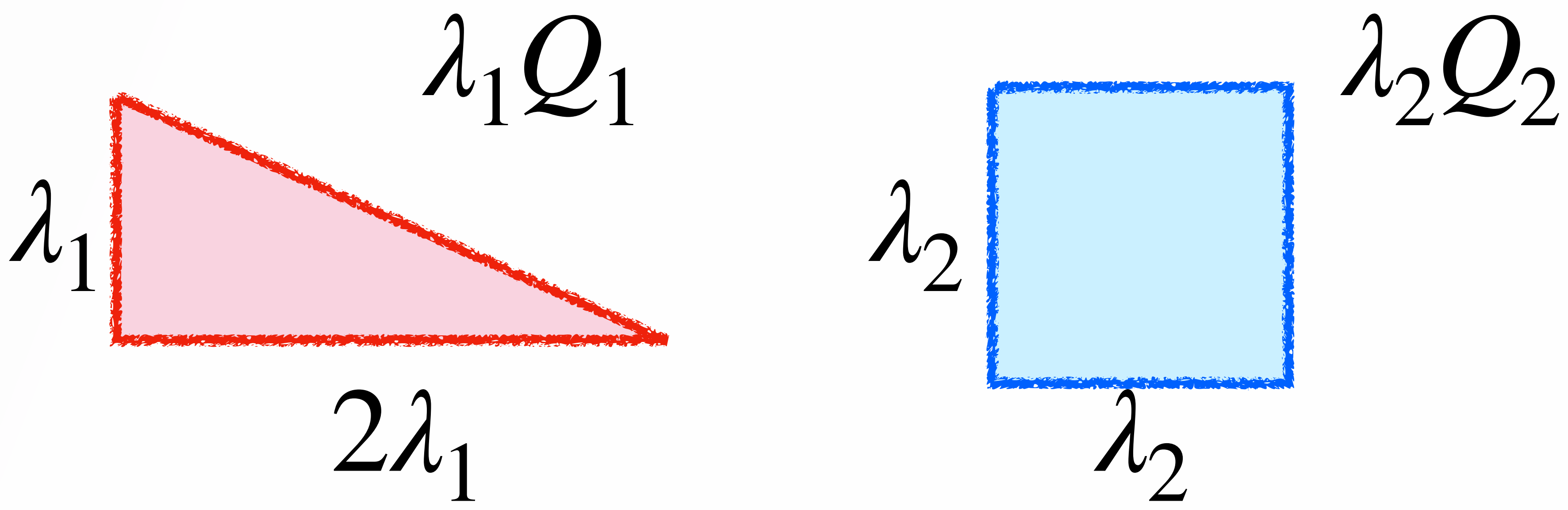
Example $f_1(x,y) = a_1x^2 + b_1x + c_1y + d_1$ and $f_2(x,y) = a_2xy + b_2x + c_2y + d_2$

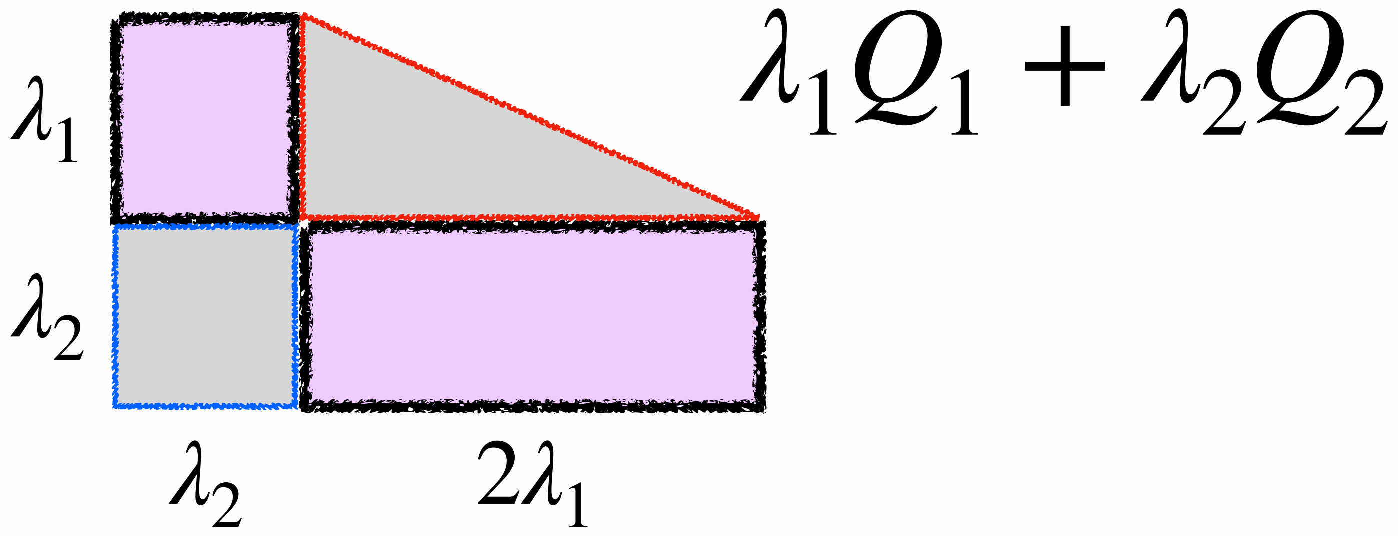
$\Rightarrow A_1 = \{(2{,}0),(1{,}0),(0{,}1),(0{,}0)\}$ and $A_2 = \{(1{,}1),(1{,}0),(0{,}1),(0{,}0)\}$

$\Rightarrow$



$\Rightarrow$





$\Rightarrow$ $\mathrm{Vol}(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_2^2$

$\Rightarrow MV(Q_1, Q_2) = 3$

Bernstein Generic System

The second part of Bernstein theorem

Theorem [Bernstein 1975]. $F := \{f_1, \ldots, f_n\} \subset \mathbb{C}[x_1, \ldots, x_n]$: a square polynomial system.

Q_i : the Newton polytope of f_i. Then, (# isolated roots in $(\mathbb{C} \setminus \{0\})^n$) $\leq MV(Q_1, \ldots, Q_n)$.

The equality holds if and only if the **facial system** $F^w := \{f_1^w, \ldots, f_n^w\}$ has no solutions in $(\mathbb{C} \setminus \{0\})^n$ for any weight vector $w \in \mathbb{Z}^n$.

When a system achieves the BKK bound, we say that the system is **Bernstein generic**.

Bernstein Generic System

The second part of Bernstein theorem

Theorem [Bernstein 1975]. $F := \{f_1, \ldots, f_n\} \subset \mathbb{C}[x_1, \ldots, x_n]$: a square polynomial system.

Q_i : the Newton polytope of f_i. Then, $(\#$ isolated roots in $(\mathbb{C}\backslash\{0\})^n) \leq MV(Q_1, \ldots, Q_n)$.
The equality holds if and only if the **facial system** $F^w := \{f_1^w, \ldots, f_n^w\}$ has no solutions in $(\mathbb{C}\backslash\{0\})^n$ for any weight vector $w \in \mathbb{Z}^n$.

When a system achieves the BKK bound, we say that the system is **Bernstein generic**.

Bernstein Generic System

The second part of Bernstein theorem

Theorem [Bernstein 1975]. $F := \{f_1, \ldots, f_n\} \subset \mathbb{C}[x_1, \ldots, x_n]$: a square polynomial system.

Q_i : the Newton polytope of f_i. Then, $(\#$ isolated roots in $(\mathbb{C}\backslash\{0\})^n) \leq MV(Q_1, \ldots, Q_n)$.
The equality holds if and only if the **facial system** $F^w := \{f_1^w, \ldots, f_n^w\}$ has no solutions in $(\mathbb{C}\backslash\{0\})^n$ for any weight vector $w \in \mathbb{Z}^n$.

When a system achieves the BKK bound, we say that the system is **Bernstein generic**.

Bernstein Generic System

Facial system

$$f = \sum_{a \in A} c_a x^a \in \mathbb{C}[x_1, \ldots, x_n]$$

a polynomial with the support $A \subset \mathbb{Z}^n$.

$$w \in \mathbb{Z}^n$$

a weight vector.

$A^w := \{a \in A \mid \langle w, a \rangle$ achieves its minimum$\}$

the **face** of A with respect to w.

Define $f^w := \sum_{a \in A^w} c_a x^a$.

$$F^w := \{f_1^w, \ldots, f_n^w\}$$

the facial system of F with respect to w

Bernstein Generic System

Facial system

$$f = \sum_{a \in A} c_a x^a \in \mathbb{C}[x_1, \ldots, x_n]$$

a polynomial with the support $A \subset \mathbb{Z}^n$.

$$w \in \mathbb{Z}^n$$

a weight vector.

$A^w := \{a \in A \mid \langle w, a \rangle$ achieves its minimum$\}$

the **face** of A with respect to w.

Define $f^w := \sum_{a \in A^w} c_a x^a$.

$$F^w := \{f_1^w, \ldots, f_n^w\}$$

the facial system of F with respect to w

Bernstein Generic System

Facial system

$$f = \sum_{a \in A} c_a x^a \in \mathbb{C}[x_1, \ldots, x_n]$$

a polynomial with the support $A \subset \mathbb{Z}^n$.

$$w \in \mathbb{Z}^n$$

a weight vector.

$A^w := \{a \in A \mid \langle w, a \rangle$ achieves its minimum$\}$

the **face** of A with respect to w.

Define $f^w := \sum_{a \in A^w} c_a x^a$.

$$F^w := \{f_1^w, \ldots, f_n^w\}$$

the facial system of F with respect to w

Bernstein Generic System

Facial system

$$f = \sum_{a \in A} c_a x^a \in \mathbb{C}[x_1, \ldots, x_n]$$

a polynomial with the support $A \subset \mathbb{Z}^n$.

$$w \in \mathbb{Z}^n$$

a weight vector.

$$A^w := \{a \in A \mid \langle w, a \rangle \text{ achieves its minimum}\}$$

the **face** of A with respect to w.

Define $f^w := \sum_{a \in A^w} c_a x^a$.

$$F^w := \{f_1^w, \ldots, f_n^w\}$$

the facial system of F with respect to w

Bernstein Generic System

Facial system

$$f = \sum_{a \in A} c_a x^a \in \mathbb{C}[x_1, \ldots, x_n]$$

a polynomial with the support $A \subset \mathbb{Z}^n$.

$$w \in \mathbb{Z}^n$$

a weight vector.

$A^w := \{a \in A \mid \langle w, a \rangle$ achieves its minimum$\}$

the **face** of A with respect to w.

Define $f^w := \sum_{a \in A^w} c_a x^a$.

$$F^w := \{f_1^w, \ldots, f_n^w\}$$

the facial system of F with respect to w

Bernstein Generic System

Facial system

$$f = \sum_{a \in A} c_a x^a \in \mathbb{C}[x_1, \ldots, x_n]$$

a polynomial with the support $A \subset \mathbb{Z}^n$.

$$w \in \mathbb{Z}^n$$

a weight vector.

$A^w := \{a \in A \mid \langle w, a \rangle$ achieves its minimum$\}$

the **face** of A with respect to w.

Define $f^w := \sum_{a \in A^w} c_a x^a$.

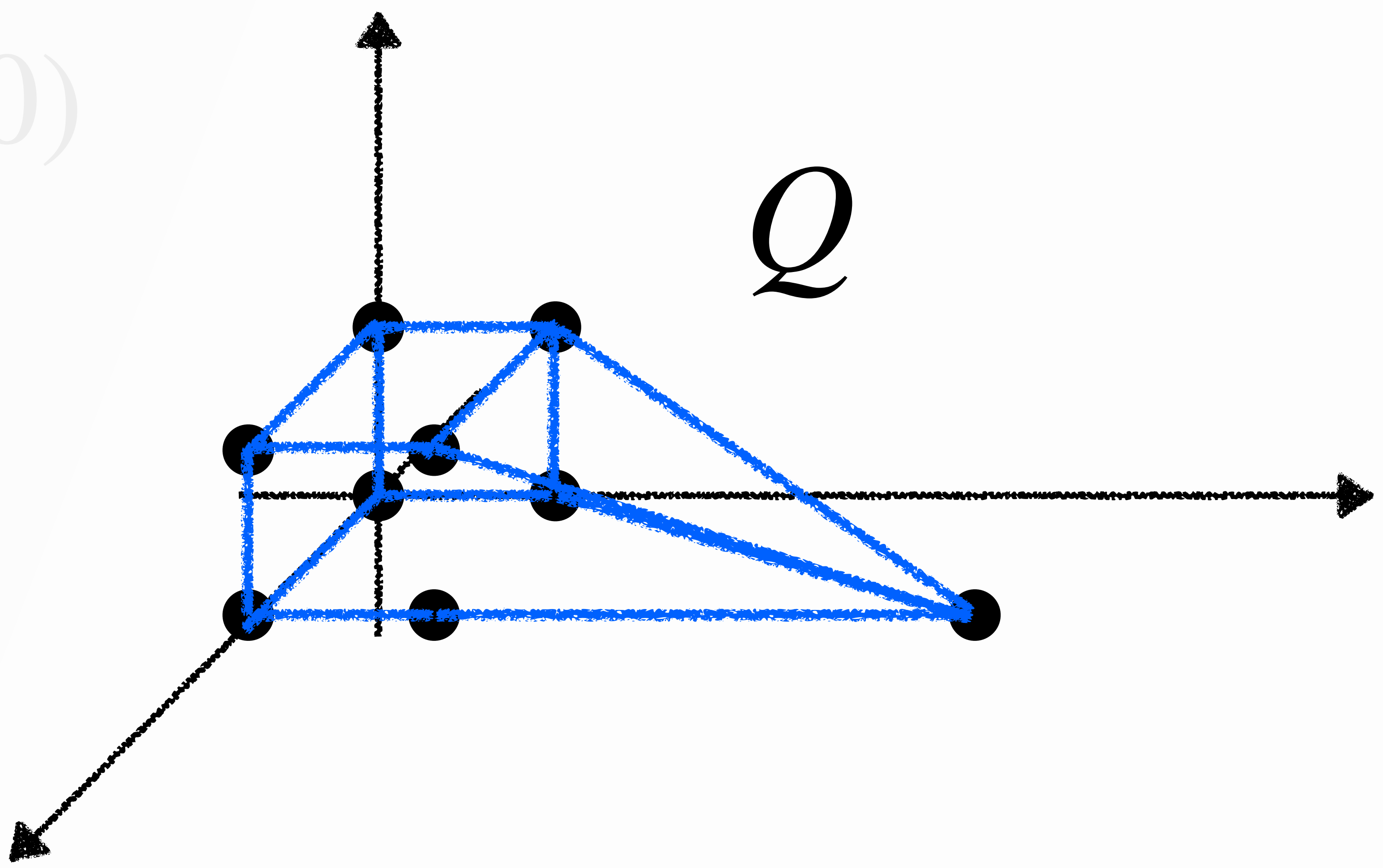
$$F^w := \{f_1^w, \ldots, f_n^w\}$$

the facial system of F with respect to w

Bernstein Generic System

Example $f = c_1xy^4 + c_2xyz + c_3xy + c_4yz + c_5xz + c_6x + c_7y + c_8z + c_9$.

$w = (-1,0,0)$

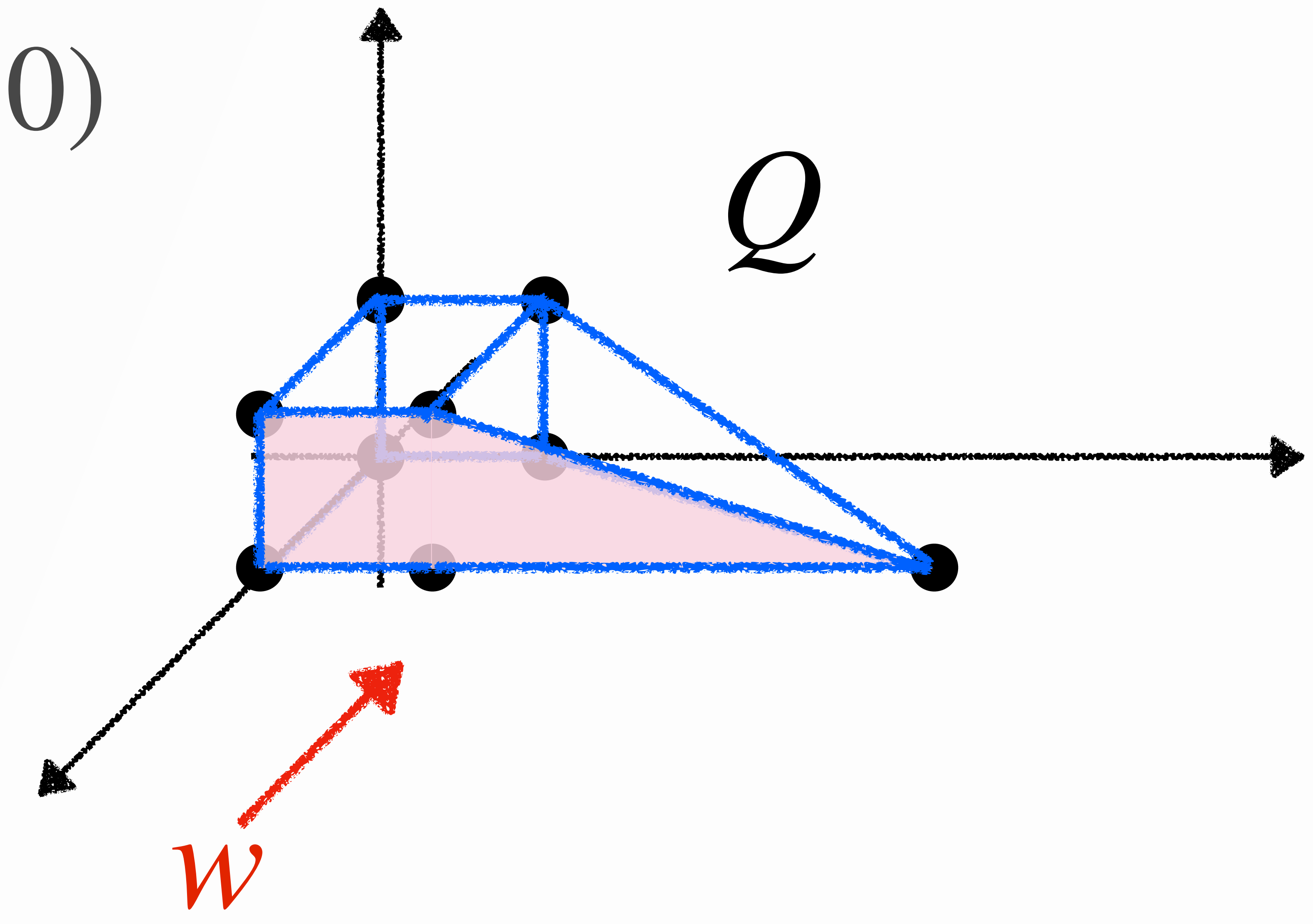


$f^w = c_1xy^4 + c_2xyz + c_3xy + c_5xz + c_6x$

Bernstein Generic System

Example $f = c_1xy^4 + c_2xyz + c_3xy + c_4yz + c_5xz + c_6x + c_7y + c_8z + c_9$.

$w = (-1,0,0)$

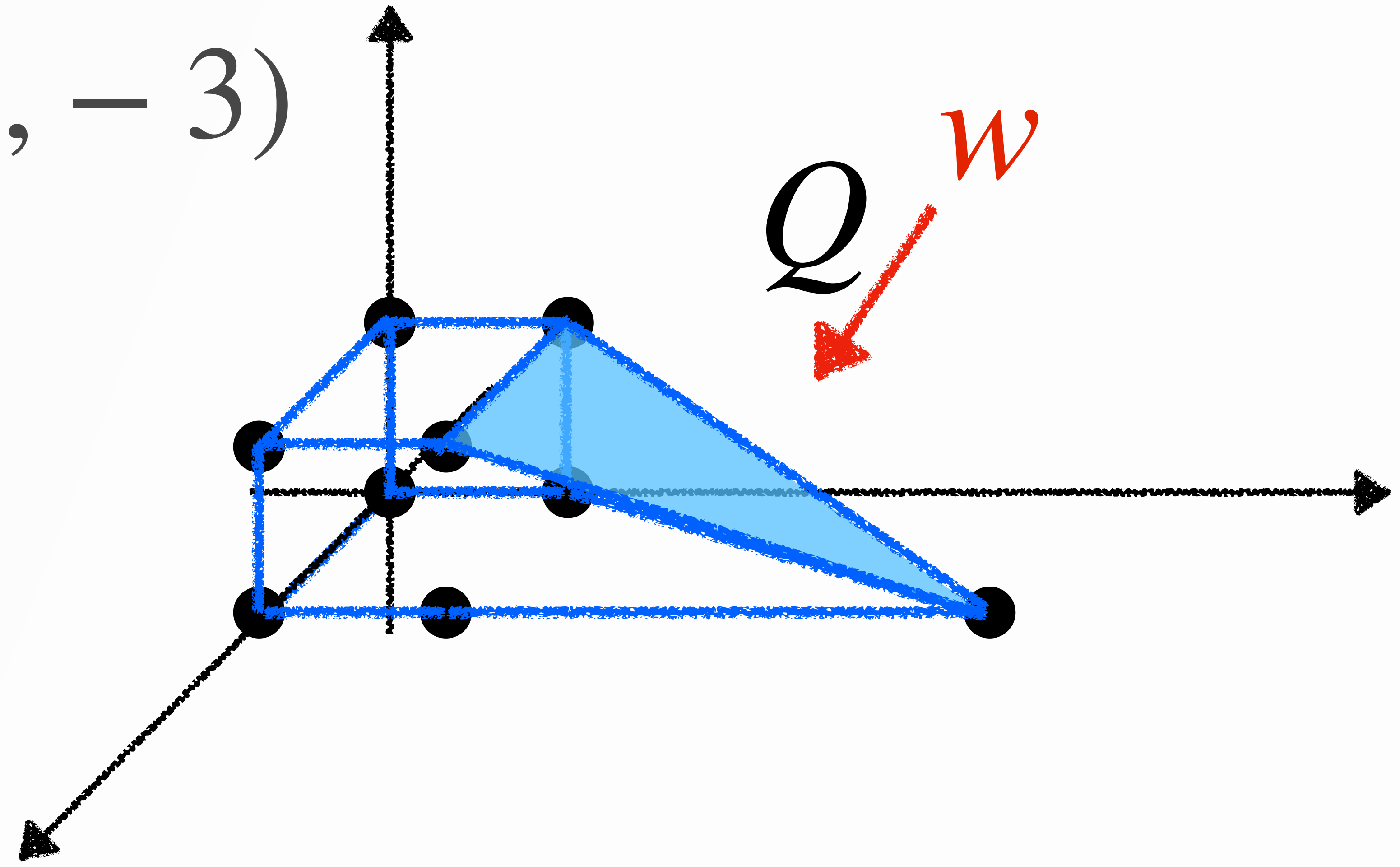


$f^w = c_1xy^4 + c_2xyz + c_3xy + c_5xz + c_6x$

Bernstein Generic System

Example $f = c_1xy^4 + c_2xyz + c_3xy + c_4yz + c_5xz + c_6x + c_7y + c_8z + c_9$.

$w = (0, -1, -3)$



$f^w = c_1xy^4 + c_2xyz + c_4yz$

Solving KKT System

Theorem [L.-Tang]. Suppose that for each $i = 1,\ldots,N$, polynomials f_i and $g_{i,j}$ are generic for all $j = 1,\ldots,m_i$. Then the KKT system $F = \{F_1, \ldots, F_N\}$ is Bernstein general.

Hence, for a generic KKT system, the polyhedral homotopy method can find all solutions.

Solving KKT System

<u>Theorem</u> [L.-Tang]. Suppose that for each $i = 1, \ldots, N$, polynomials f_i and $g_{i,j}$ are generic for all $j = 1, \ldots, m_i$. Then the KKT system $F = \{F_1, \ldots, F_N\}$ is Bernstein general.

Hence, for a generic KKT system, the polyhedral homotopy method can find all solutions.

Selecting GNEs from KKT Points

$\mathscr{K}_{\mathbb{C}}$: the set of KKT points (x, λ) obtained by the homotopy method.

$\mathscr{K}$: the set of real KKT points with $\lambda \geq 0$ and $g_{i,j}(x) \geq 0$.

$$\mathscr{K} = \{(x, \lambda) \in \mathscr{K}_{\mathbb{C}} \cap \mathbb{R}^N \mid \lambda_{i,j} \geq 0,\ g_{i,j}(x) \geq 0\}$$

$\mathscr{P} := \pi_x(\mathscr{K})$: a projection of $\mathscr{K}$ onto x coordinates.

Selecting GNEs from KKT Points

$\mathcal{K}_{\mathbb{C}}$: the set of KKT points (x, λ) obtained by the homotopy method.

$\mathcal{K}$: the set of real KKT points with $\lambda \geq 0$ and $g_{i,j}(x) \geq 0$.

$$\mathcal{K} = \{(x, \lambda) \in \mathcal{K}_{\mathbb{C}} \cap \mathbb{R}^N \mid \lambda_{i,j} \geq 0,\ g_{i,j}(x) \geq 0\}$$

$\mathcal{P} := \pi_x(\mathcal{K})$: a projection of $\mathcal{K}$ onto x coordinates.

Selecting GNEs from KKT Points

$\mathscr{K}_{\mathbb{C}}$: the set of KKT points (x, λ) obtained by the homotopy method.

$\mathscr{K}$: the set of real KKT points with $\lambda \geq 0$ and $g_{i,j}(x) \geq 0$.

$$\mathscr{K} = \{(x, \lambda) \in \mathscr{K}_{\mathbb{C}} \cap \mathbb{R}^N \mid \lambda_{i,j} \geq 0,\ g_{i,j}(x) \geq 0\}$$

$\mathscr{P} := \pi_x(\mathscr{K})$: a projection of $\mathscr{K}$ onto x coordinates.

Selecting GNEs from KKT Points

For $u = (u_1, \ldots, u_N) \in \mathscr{P}$, consider the following optimization problem:

$$\begin{cases} \delta_i := \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, u_{-i}) - f_i(u_i, u_{-i}) \\ \qquad\quad \text{s.t.} & g_{i,j}(x_i, u_{-i}) = 0 \quad \text{if } j \in \mathscr{E}_i \\ & g_{i,j}(x_i, u_{-i}) \geq 0 \quad \text{if } j \in \mathscr{I}_i \end{cases}$$

If u is a GNE, then each u_i is a minimizer.

We solve the optimization problem using the moment-SOS relaxation.

Experiments

Example 1) Non-convex Problem

$$\text{1st player}: \begin{cases} \min\limits_{x_1 \in \mathbb{R}^2} & 3x_{2,1}(x_{1,1})^3 + 5(x_{1,2})^3 - 2\sum_{j=1}^{2} x_{1,j} \cdot \sum_{j=1}^{2} x_{2,j} \\ s.t. & 5x_{1,1} - 2x_{1,2} + 3x_{2,2} - 1 \geq 0,\ 3 - x_{2,1} \cdot x_1^T x_1 \geq 0, \\ & x_{1,1} \geq -2,\ x_{1,2} \geq 1; \end{cases} \quad \text{2nd player}: \begin{cases} \min\limits_{x_2 \in \mathbb{R}^2} & (2x_{1,1} + 3x_{1,2})(x_{2,1})^3 - 3x_{2,1} + 7(x_{2,2})^2 + 5x_{1,1}x_{1,2}x_{2,2} \\ s.t. & 7x_{1,2} + 3x_{2,2} - 5x_{2,1}^2 + 3 \geq 0,\ 2x_{2,1} \geq -1, \\ & 2 - x_{2,2} \geq 0,\ 5 + x_{2,2} \geq 0. \end{cases}$$

The KKT system has the mixed volume 480, Solving using `HomotopyContinuation.jl`, it 480 KKT points and gives a unique GNE in 5.75 seconds (4 seconds to compute KKT points, 1.75 seconds for selecting).

$$(0.7636,\ 1,\ 0.47,\ -0.2727)$$

Experiments

Example 1) Non-convex Problem

$$\text{1st player}: \begin{cases} \min\limits_{x_1 \in \mathbb{R}^2} & 3x_{2,1}(x_{1,1})^3 + 5(x_{1,2})^3 - 2\sum_{j=1}^{2} x_{1,j} \cdot \sum_{j=1}^{2} x_{2,j} \\ s.t. & 5x_{1,1} - 2x_{1,2} + 3x_{2,2} - 1 \geq 0,\ 3 - x_{2,1} \cdot x_1^T x_1 \geq 0, \\ & x_{1,1} \geq -2,\ x_{1,2} \geq 1; \end{cases} \quad \text{2nd player}: \begin{cases} \min\limits_{x_2 \in \mathbb{R}^2} & (2x_{1,1} + 3x_{1,2})(x_{2,1})^3 - 3x_{2,1} + 7(x_{2,2})^2 + 5x_{1,1}x_{1,2}x_{2,2} \\ s.t. & 7x_{1,2} + 3x_{2,2} - 5x_{2,1}^2 + 3 \geq 0,\ 2x_{2,1} \geq -1, \\ & 2 - x_{2,2} \geq 0,\ 5 + x_{2,2} \geq 0. \end{cases}$$

The KKT system has the mixed volume 480, Solving using `HomotopyContinuation.jl`, it 480 KKT points and gives a unique GNE in 5.75 seconds (4 seconds to compute KKT points, 1.75 seconds for selecting).

$$(0.7636,\ 1,\ 0.47,\ -0.2727)$$

Experiments

Example 2) Convex Problem

Example A.3 of **[Facchinei-Kanzow 2010]**

A 3-player game with objectives $f_i = \frac{1}{2}x_i^\top A_i x_i + x_i^\top(B_i x_{-i} + b_i)$ where

$$A_1 = \begin{bmatrix} 20 & 5 & 3 \\ 5 & 5 & -5 \\ 3 & -5 & 15 \end{bmatrix}, A_2 = \begin{bmatrix} 11 & -1 \\ -1 & 9 \end{bmatrix}, A_3 = \begin{bmatrix} 48 & 39 \\ 39 & 53 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -6 & 10 & 11 & 20 \\ 10 & -4 & -17 & 9 \\ 15 & 8 & -22 & 21 \end{bmatrix}, B_2 = \begin{bmatrix} 20 & 1 & -3 & 12 & 1 \\ 10 & -4 & 8 & 16 & 21 \end{bmatrix}, B_3 = \begin{bmatrix} 10 & -2 & 22 & 12 & 16 \\ 9 & 19 & 21 & -4 & 20 \end{bmatrix}, b_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Constraints are given $-10 \le x \le 10,\ g_{1,1} = 20 - x_{1,1} - x_{1,2} - x_{1,3} \ge 0,\ g_{1,2} = x_{2,1} - x_{3,2} - x_{1,1} - x_{1,2} + x_{1,3} + 5 \ge 0,$
$g_{2,1} = x_{1,2} + x_{1,3} - x_{3,1} - x_{2,1} + x_{2,2} + 7 \ge 0,\ g_{3,1} = x_{1,1} + x_{1,3} - x_{2,1} - x_{3,2} + 4 \ge 0.$

Experiments

Example 2) Convex Problem

Example A.3 of **[Facchinei-Kanzow 2010]**

The mixed volume : 12096

Solution found : 11631 KKT points

GNE found : 5 GNEs found with 4 newly found

Elapsed time : 177 seconds

Experiments

Comparison

Comparison with known methods on Example 1) and 2) :

Interior point method **(Dreves-Facchinei-Kanzow-Sagratella 2011)**

Augmented Lagrangian method **(Kanzow-Steck 2016)**

Gauss-Seidel method **(Nie-Tang-Xu 2021)**

Semidefinite relaxation **(Nie-Tang 2021)**

Experiments

Comparison

IPM : Interior point method, **ALM** : Augmented Lagrangian method, **GSM** : Gauss-Seidel method, **SDP** : Semidefinite relaxation,

PHC : Solved by using the polyhedral homotopy method (`HomotopyContinuation.jl`)

		IPM	ALM	GSM	SDP	PHC
Example 1)	**Time**	Fail	Fail	11.47	17.89	5.75
	Error			$4 \cdot 10^{-7}$	$1 \cdot 10^{-6}$	$2 \cdot 10^{-8}$
Example 2)	**Time**	3.12	1.50	Fail	11.55	177
	Error	$2 \cdot 10^{-7}$	$1 \cdot 10^{-7}$		$2 \cdot 10^{-7}$	$1 \cdot 10^{-6}$ (5 GNEs)

Experiments

Example 3) Random nonconvex GNEP

Consider N-player GNEP whose i-th player's optimization problem is

$$\begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, x_{-i}) \\ \text{s.t.} & -x_i^\top A_i x_i + x_{-i}^\top B_i x_i + c_i^\top x \geq d_i \end{cases}$$

where $A_i = R_i^\top R_i$ with randomly generated $R_i \in \mathbb{R}^{n_i \times n_i}$ and $B_i \in \mathbb{R}^{n_i \times (n-n_i)}, c_i \in \mathbb{R}^n, d_i \in \mathbb{R}$.

The objective f_i is a dense polynomial of degree d with randomly generated real coefficients.

For various (d, N, n_i)-values, solve the problem 100 times and record the success rate (for finding mixed volume many KKT points) and elapsed time (solving KKT + selecting GNEs).

Experiments

Example 3) Random nonconvex GNEP

Consider N-player GNEP whose i-th player's optimization problem is

$$\begin{cases} \min\limits_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, x_{-i}) \\ \text{s.t.} & -x_i^\top A_i x_i + x_{-i}^\top B_i x_i + c_i^\top x \geq d_i \end{cases}$$

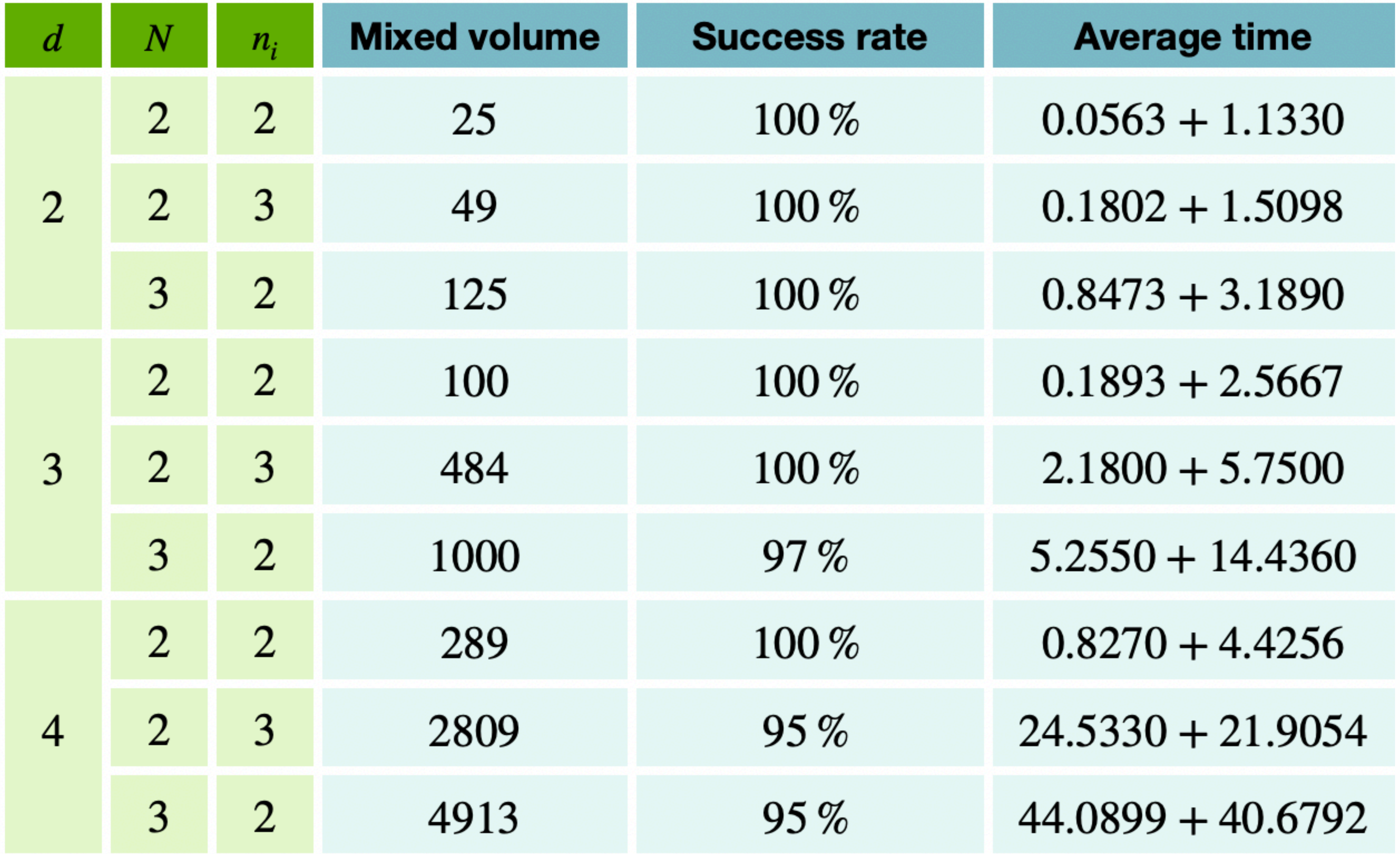
where $A_i = R_i^\top R_i$ with randomly generated $R_i \in \mathbb{R}^{n_i \times n_i}$ and $B_i \in \mathbb{R}^{n_i \times (n-n_i)}, c_i \in \mathbb{R}^n, d_i \in \mathbb{R}$.

The objective f_i is a dense polynomial of degree d with randomly generated real coefficients.

For various (d, N, n_i)-values, solve the problem 100 times and record the success rate (for finding mixed volume many KKT points) and elapsed time (solving KKT + selecting GNEs).

Experiments

Example 3) Random nonconvex GNEP



d	N	n_i	Mixed volume	Success rate	Average time
2	2	2	25	100 %	$0.0563 + 1.1330$
	2	3	49	100 %	$0.1802 + 1.5098$
	3	2	125	100 %	$0.8473 + 3.1890$
3	2	2	100	100 %	$0.1893 + 2.5667$
	2	3	484	100 %	$2.1800 + 5.7500$
	3	2	1000	97 %	$5.2550 + 14.4360$
4	2	2	289	100 %	$0.8270 + 4.4256$
	2	3	2809	95 %	$24.5330 + 21.9054$
	3	2	4913	95 %	$44.0899 + 40.6792$

Thank you for your attention