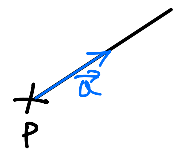
Recall $P(x_0, y_0, z_0)$

$\vec{r}_0 = \vec{OP}$, $\vec{a} = (a, b, c)$

(direction of the line)



$\vec{r} = \vec{OQ}$ (Q: a point on the line)

$\vec{r} = \vec{r}_0 + t\vec{a}$

(vector equation)

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

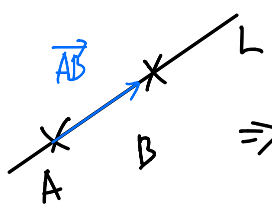
(parametric equations)

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

(symmetric equations)

* converting from symmetric equations to vector equations

Let $A(x_0, y_0, z_0)$, $B(x_1, y_1, z_1)$



$\vec{AB} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

$$\Rightarrow \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0} \; (= t) \quad \text{(symmetric equations)}$$

$$\Rightarrow \begin{cases} x - x_0 = t(x_1 - x_0) \\ y - y_0 = t(y_1 - y_0) \\ z - z_0 = t(z_1 - z_0) \end{cases} \Rightarrow (x - x_0, y - y_0, z - z_0) = t(x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

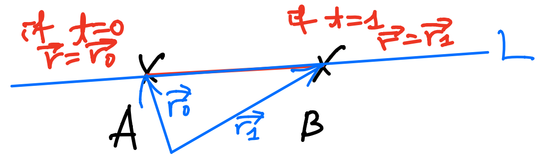
Let $\vec{r}_0 = \vec{OA}$, $\vec{r}_1 = \vec{OB}$ & $\vec{r} = (x, y, z)$

$\Rightarrow \vec{r} - \vec{r}_0 = t(\vec{r}_1 - \vec{r}_0)$

$\Rightarrow \vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = (1 - t)\vec{r}_0 + t\vec{r}_1$

(vector equation)

* line segment.



$A(x_0, y_0, z_0)$, $\quad B(x_1, y_1, z_1)$

Let $\vec{r_0} = \overrightarrow{OA}$, $\vec{r_1} = \overrightarrow{OB}$ $\Rightarrow$ $\vec{r} = (1-t)\vec{r_0} + t\vec{r_1}$

(vector equation for L)

$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1}$ for $0 \le t \le 1$

: line segment equation.

Example) Let $L_1 = \begin{cases} x = 1+t \\ y = -2+3t \\ z = 4-t \end{cases}$ & $L_2 = \begin{cases} x = 2s \\ y = 3+s \\ z = -3+4s \end{cases}$

Show that they are skew (neither intersect nor parallel)

① Checking not parallel

$\begin{cases} (\text{direction number for } L_1) = (1, 3, -1) \\ (\text{direction number for } L_2) = (2, 1, 4) \end{cases}$ not parallel.

② Checking there is no intersection.

$\begin{cases} 1+t = 2s \\ -2+3t = 3+s \\ 4-t = -3+4s \end{cases}$ for some t & s if they intersect

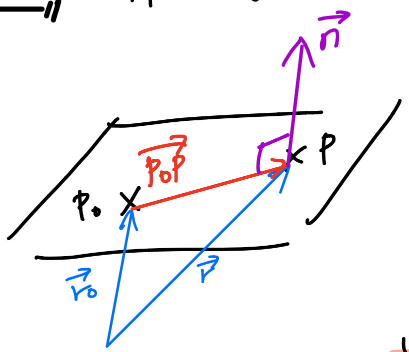
$\Rightarrow \begin{cases} t = 2s-1 \\ 3t = 5+s \\ t = 7-4s \end{cases} \Rightarrow \begin{cases} 2s-1 = 7-4s \\ 3(2s-1) = 5+s \end{cases} \Rightarrow \begin{cases} 6s = 8 \\ 5s = 8 \end{cases}$

$\Rightarrow$ no such s $\Rightarrow$ no intersection.

* Planes

A vector perpendicular to a plane can be determined

Recall// if $\vec{a}$, $\vec{b}$ are perpendicular, $\vec{a}\cdot\vec{b}=0$



P, P_0 : given points on the plane

$\vec{r}_0 = \overrightarrow{OP_0}$, $\vec{r} = \overrightarrow{OP}$

$\Rightarrow \overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \vec{r} - \vec{r_0}$

vector equation of the plane

$\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0.$

Represent a vector equation into components

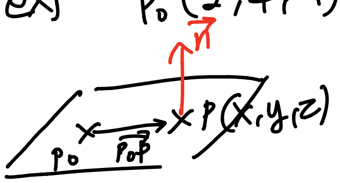
$\vec{n} = (a,b,c)$, $\vec{r_0} = (x_0, y_0, z_0)$, $\vec{r} = (x, y, z)$

(normal vector)

$(a,b,c)\cdot(x-x_0,\ y-y_0,\ z-z_0) = a(x-x_0) + b(y-y_0) + c(z-z_0)$
$= 0$

(scalar equation of the plane).

ex) $P_0(2,4,-1)$, $\vec{n} = (2,3,4)$.



$\Rightarrow (2,3,4)\cdot\underbrace{(x-2,\ y-4,\ z+1)}_{\overrightarrow{P_0P}}$

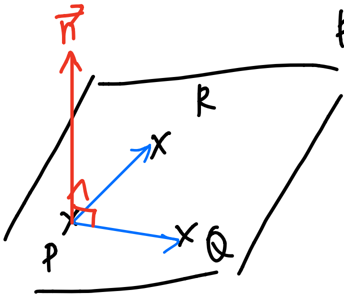
$= 2(x-2) + 3(y-4) + 4(z+1) = 0.$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$: a scalar equation of a plane.

$\Rightarrow ax + by + cz + d = 0$ where $d = -ax_0 - by_0 - cz_0$

linear equation of the plane

Example) Find a linear equation of the plane passing thru $P(1,2,3)$, $Q(3,-2,7)$, $R(5,1,1)$



Find $\vec{n}$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (3,-2,7) - (1,2,3)$$
$$= (2,-4,4)$$
$$\vec{PR} = \vec{OR} - \vec{OP} = (5,1,1) - (1,2,3)$$
$$= (4,-1,-2).$$

$$\Rightarrow \vec{n} = \vec{PQ} \times \vec{PR} = \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix}$$

$$= \vec{i}(12) - \vec{j}(-20) + \vec{k}(14) = 12\vec{i} + 20\vec{j} + 14\vec{k}$$

$$\Rightarrow 12(x-1) + 20(y-2) + 14(z-3) = 0$$

$$\Rightarrow 12x + 20y + 14z - 12 - 40 - 42 = 0$$

$$\Rightarrow 12x + 20y + 14z = 94 \Rightarrow 6x + 10y + 7z = 47.$$