Chapter 8. Differential Equations.

※ terminologies.

Differential Equation : an equation containing derivatives.

ex) $\frac{dy}{dt} = ry(t)$.

order : the highest order of derivative in a d.e.

ex) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = xy$: Second order.

$\frac{dy}{dx} = y(x)$: first order.

Solution : a function satisfying a given d.e.

Separable equation : a d.e with a form

$$\frac{dy}{dx} = f(x)g(y).$$

derivative only

multiplication of functions of x, y.

* Pure - Time Differential Equations.

$$\frac{dy}{dx} = f(x) \quad \text{for } x \text{ in } I.$$

$$\Rightarrow \quad y(x) = \int_{x_0}^{x} f(u)\,du + C \qquad \text{by FTC.}$$

let $y(x_0) = y_0$

$$\Rightarrow \quad y(x) = y_0 + \int_{x_0}^{x} f(u)\,du.$$

Example). $\frac{dv}{dt} = \sin t, \qquad V(0) = 3.$

Find $v(t)$

$$V(t) = V(0) + \int_0^t \sin x\,dx$$

$$= 3 + [-\cos x]_0^t = 3 + [-\cos t + 1]$$

$$= 4 - \cos t.$$

* Autonomous Differential Equations.

(Many of biological situations have this model)

$$\frac{dy}{dx} = g(y).$$

ex). $\frac{dN}{dt} = 2N(t)$. : a population model.

Consider two different initial value conditions with the same size.

$N(0) = 20$] 2 different initial values with the same size.
$N(10) = 20$]

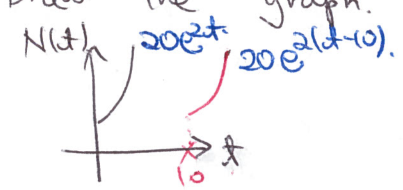
the solution is $N(t) = Ce^{2t}$.

if $N(0) = 20 \Rightarrow N_1(t) = 20e^{2t}$.

if $N(10) = 20 \Rightarrow 20 = Ce^{20} \Rightarrow C = 20e^{-20}$.

$\Rightarrow N_2(t) = 20e^{2(t-10)}$.

Draw the graph.



→ Same graph with different starting points.
(Same result regardless of when we start.)

autonomous.

Example) $\frac{dy}{dx} = 2-3y, \quad y(1) = 1.$

$$\frac{dy}{2-3y} = dx. \rightarrow -\frac{1}{3}\ln|2-3y| = x + C$$

Solve for y.

$$\ln|2-3y| = -3x + C.$$

$$|2-3y| = e^{-3x+C} = e^{-3x}\cdot e^{C} = Ce^{-3x}.$$

$$\Rightarrow 2-3y = \underbrace{C}_{C=\pm e^{-3C_1}}e^{-3x}.$$

$$y(1)=1 \Rightarrow 2-3 = Ce^{-3} \Rightarrow C = -e^{3}.$$

$$\Rightarrow 2-3y = -e^{3}e^{-3x} \Rightarrow y = \frac{2}{3} + \frac{1}{3}e^{3-3x}.$$

Example) (Exponential Growth)

$$\frac{dN}{dt} = rN, \quad N(0) = N_0, \quad r>0$$

$$\frac{1}{rN}dN = dt \Rightarrow \frac{1}{r}\ln|N| = t + C. \Rightarrow \ln|N| = rt + C$$

$$\Rightarrow |N| = e^{rt}\cdot e^{C} \Rightarrow N = \pm e^{C}\cdot e^{rt} = C\cdot e^{rt}.$$

$$\xRightarrow{N(0)=N_0} N_0 = C\cdot 1 \Rightarrow C = N_0 \Rightarrow N(t) = N_0 e^{rt}.$$

Example) (Restricted Growth)

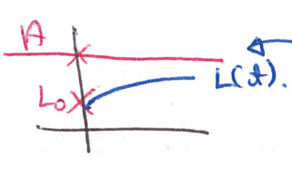
$$\frac{dL}{dt} = k(A-L), \quad L(0) = L_0 \quad \text{where} \quad 0 < L_0 < A, \quad k>0.$$

$$\frac{-1}{k(A-L)}dL = dt \Rightarrow -\frac{1}{k}\ln|A-L| = t + C \Rightarrow -\ln|A-L| = kt + C.$$

$$\Rightarrow |A-L| = e^{-kt}\cdot e^{c} \Rightarrow A-L = \pm e^{c}\cdot e^{-kt} = Ce^{-kt}.$$

$$\Rightarrow -L = -A + C\cdot e^{-kt} \xrightarrow{L(0)=L_0} -L_0 = -A + C \Rightarrow C = A - L_0.$$

$$\Rightarrow -L = -A + (A-L_0)e^{-kt} \Rightarrow L = A + \underbrace{(L_0-A)}_{<0}\underbrace{e^{-kt}}_{\text{decreasing}}.$$



bounded
restricted.

Example) $\frac{dy}{dx} = 2(y-1)(y+2)$, $y(0) = 2$.

$$\frac{1}{2(y-1)(y+2)} dy = dx.$$

partial fraction.

$$\frac{1}{(y-1)(y+2)} = \frac{A}{y-1} + \frac{B}{y+2}.$$

$$\Rightarrow 1 = A(y+2) + B(y-1) = (A+B)y + 2A - B.$$

$$\begin{cases} A+B=0 \\ 2A-B=1 \end{cases} \Rightarrow A = \frac{1}{3}, \quad B = -\frac{1}{3}.$$

$$\frac{1}{2}\left[\frac{1}{3(y-1)} - \frac{1}{3(y+2)}\right] dy = dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{3}\ln|y-1| - \frac{1}{3}\ln|y+2|\right] = x + C.$$

$$\Rightarrow \ln|y-1| - \ln|y+2| = 6x + C.$$

$$\Rightarrow \ln\left|\frac{y-1}{y+2}\right| = 6x + C \Rightarrow \frac{y-1}{y+2} = e^{6x} \cdot e^{C} = C \cdot e^{6x}.$$

$$\xRightarrow{y(0)=2} \frac{2-1}{2+2} = C \cdot 1 \Rightarrow C = \frac{1}{4}. \Rightarrow \frac{y-1}{y+2} = \frac{1}{4}e^{6x}.$$

$$\Rightarrow y-1 = \frac{1}{4}e^{6x}(y+2) \Rightarrow y - \frac{1}{4}e^{6x}y = \frac{1}{2}e^{6x} + 1$$

$$\Rightarrow y = \frac{\frac{1}{2}e^{6x}+1}{1-\frac{1}{4}e^{6x}} = \frac{2e^{6x}+4}{4-e^{6x}}.$$

Example) (Logistic Equation)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0.$$

* Allometric Growth.

Example) $\dfrac{dy}{dx} = \dfrac{y+1}{x}$, $y(1) = 0$.

$$\frac{1}{y+1}\,dy = \frac{1}{x}\,dx \Rightarrow \ln|y+1| = \ln|x| + C.$$

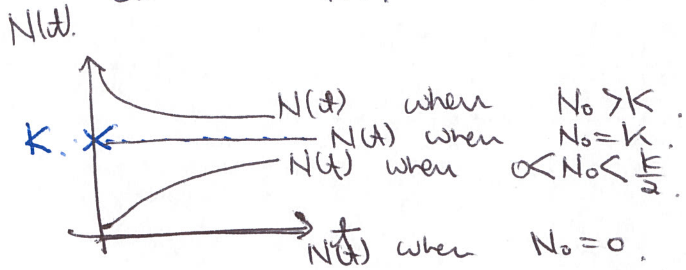
$$\Rightarrow |y+1| = |x|e^{C} = C|x|. \Rightarrow y+1 = \pm Cx. \Rightarrow y = Cx - 1.$$

$$\xrightarrow{y(1)=0} 0 = C - 1 \Rightarrow C = 1. \Rightarrow y = x - 1.$$

§8.2. Equilibria and Stability.

From now on, we only deal with autonomous eq, with t as time.

Motivation $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$ (logistic growth).



→ graph depends on N_0. (initial value)

→. long-term behavior depends on N_0.
($t \to \infty$).

* point equilibria

: constant solutions for autonomous differential equation.

ex) $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$.

if $N_0 = 0 \Rightarrow N(t) = 0$.

$N_0 = K \Rightarrow N(t) = K$

0, K are point equilibria.

* Stability for point equilibria.

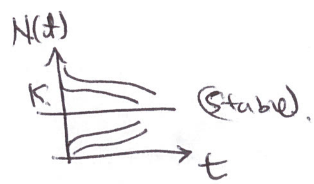
$\hat{y}$: an equilibrium for $\frac{dy}{dx} = g(y)$.

($g(\hat{y}) = 0$).

$\hat{y}$: Stable if the solution around $\hat{y}$ returns to $\hat{y}$. as $t \to \infty$.

(eventhough we change initial condition slightly, It will converge when $t \to \infty$).

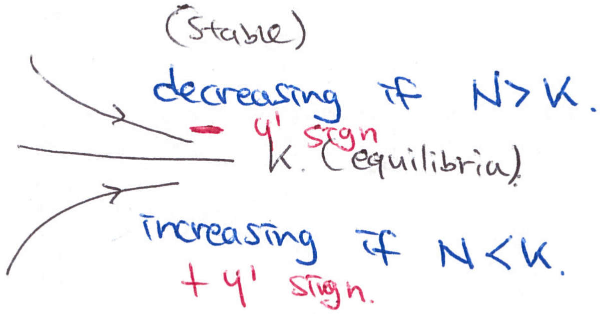
$\hat{y}$: unstable if the solution around $\hat{y}$ does not returns to $\hat{y}$ as $t \to \infty$.

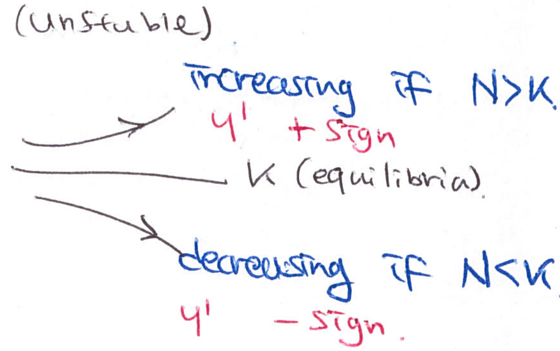
ex) 

Q: How to find the stability for $\frac{dy}{dt} = g(y)$?

$\longrightarrow$ Consider the sign of $g(y)$!

Remark



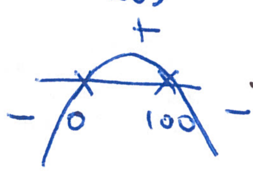


$$\frac{dy}{dt} = g(y)$$

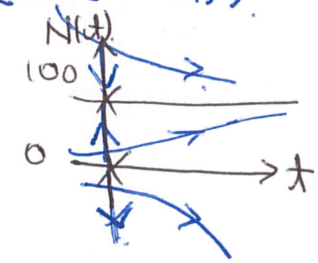
(Sign of $g(y)$) = (Sign of y').

Example) $\frac{dN}{dt} = 2N\left(1 - \frac{N}{100}\right)$

$2N\left(1-\frac{N}{100}\right) \Rightarrow \frac{1}{100}\left(2N(100-N)\right).$



$\longrightarrow$



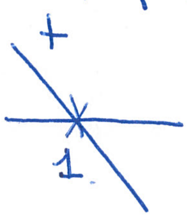
(Stable)

(Unstable)

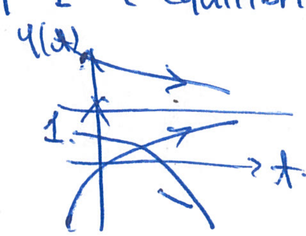
Example) Find equilibria and determine the stability.

$$\frac{dy}{dx} = 1 - y$$

$1 - y = 0 \Rightarrow y = 1$ (equilibrium)



$\Rightarrow$



(stable).

Example) (The Levins Model).

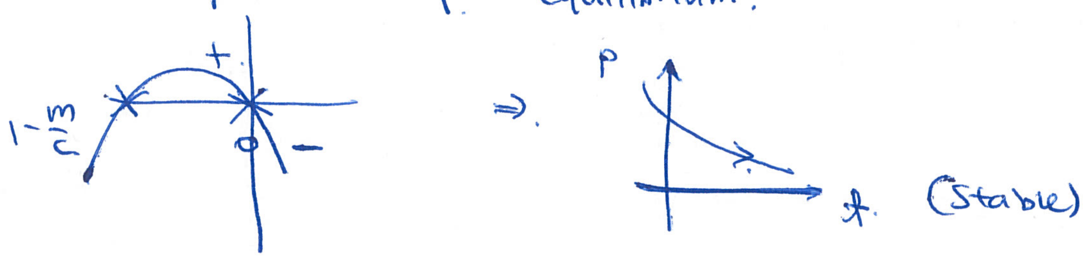
$$\frac{dp}{dt} = cp(1-p) - mp \; .$$

$$cp\left(1 - \frac{m}{c} - p\right) = 0.$$

$p = 0$ or $1 - \frac{m}{c}$.

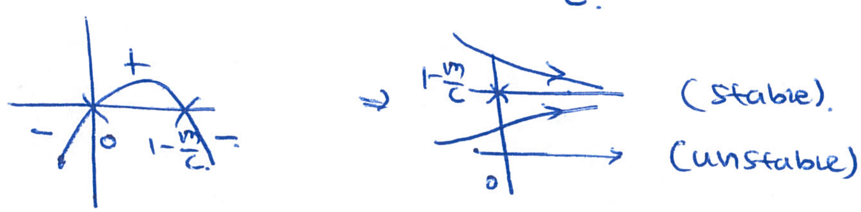
i). if $m > c$. then $1 - \frac{m}{c} < 0$.

$\Rightarrow$ $p = 0$. only equilibrium.



ii) if $m < c$. then $1 - \frac{m}{c} > 0$.

$\Rightarrow$ $p = 0$ & $p = 1 - \frac{m}{c}$.



Chapter 10. Multivariable Calculus.

* Real valued Functions.

ex) $f(x) = \sqrt{x}$ on $[0,4]$ $\xrightarrow{\text{function notation.}}$ $f: \underbrace{[0,4]}_{\text{Domain.}} \to \mathbb{R}$ Codomain.

$x \mapsto \sqrt{x}$. range $[0,2]$.

↓ n-dimension.

$f: D \to \mathbb{R}$. D in $\mathbb{R}^n$.

domain

$(x_1, \cdots, x_n) \mapsto f(x_1, \cdots, x_n)$.

$\{ w \text{ in } \mathbb{R} \mid f(x_1, \cdots, x_n) = w \text{ for } (x_1, \cdots, x_n) \text{ in } D \}$.

: the range of f.

Example). $f(x,y,z) = \dfrac{xy}{z^2}$.

Find $f(2,3,-1)$, $f(-1,2,3)$

$f(2,3,-1) = \dfrac{2 \cdot 3}{(-1)^2}$ → gives a real number.

(domain in 3-dimension)

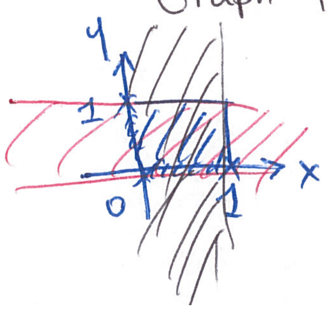
$f(-1,2,3) = \dfrac{(-1) \cdot 2}{3^2} = -\dfrac{2}{9}$.

Example). $D = \{ (x,y) \mid 0 \le x \le 1, \ 0 \le y \le 1 \}$.

$f: D \to \mathbb{R}$

$(x,y) \mapsto x+y$.

Graph the domain and find the range of f.

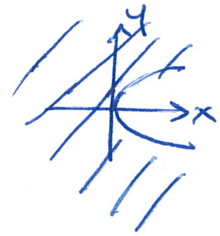


$0 \le x \le 1$
$0 \le y \le 1$
$0 \le x+y \le 2$.

$\Rightarrow \{ z \text{ in } \mathbb{R} \mid 0 \le z \le 2 \}$.

Example) Find the largest possible domain of $f(x,y) = \sqrt{y^2 - x}$.

$y^2 - x \geq 0 \Rightarrow y^2 \geq x$



* Graph of a function of Two Variables.

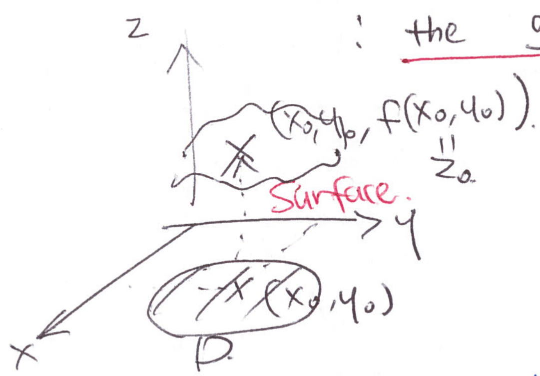
D in $\mathbb{R}^2$.

$\Rightarrow \quad f : D \to \mathbb{R}$

$\quad (x,y) \mapsto z$

$\Rightarrow \{ (x,y,z) \text{ in } \mathbb{R}^3 \mid f(x,y) = z \}$

: the graph of f



Collect the same value of $f(x,y)$.

$(f(x,y) = c)$

$\Rightarrow$. Curve in $\mathbb{R}^3$.

(level curve).

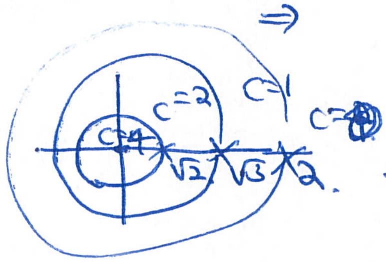
Example) $D = \{ (x,y) \mid x^2+y^2 \le 4 \}$.

$f(x,y) = 4-x^2-y^2$.

$g(x,y) = \sqrt{4-x^2-y^2}$.

$f(x,y)$
$= 4-x^2-y^2 = c \Rightarrow x^2+y^2 = 4-c$.

$\Rightarrow 4-c \ge 0 \Rightarrow 4 \ge c$.

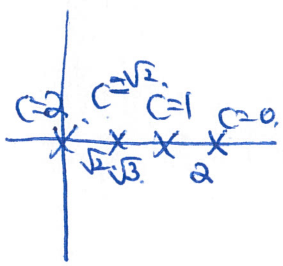


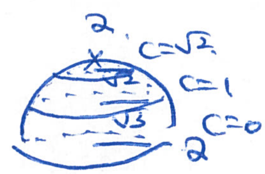
$\Rightarrow$.

$g(x,y) = \sqrt{4-x^2-y^2} = c \Rightarrow 4-x^2-y^2 = c^2$.

$\Rightarrow x^2+y^2 = 4-c^2$.

$\Rightarrow 4-c^2 \ge 0 \Rightarrow 4 \ge c^2 \Rightarrow -2 \le c \le 2$.





~~§10.2. Limits and Continuity.~~

§10.3. Partial Derivatives.

Motivation (1969) Bisek.

the net assimilation of CO_2.

$\rightarrow$ { vary the temparature.
light intensity constant.

$\rightarrow$ { consistent temparature
different light intensity.

$f(x,y)$: function of the assimilation of CO_2.

x: temperature

y: light intensity.

Q. how $f(x,y)$ changes when x and y change?

* Partial Derivative.

f: a function of x and y.

$$\Rightarrow \frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

Likewise

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

$$f_x(x,y) = \frac{\partial f}{\partial x}. \qquad f_y = \frac{\partial f}{\partial y}.$$

Example) $f(x,y) = ye^{xy}$. Find $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$.

$$\frac{\partial f}{\partial x} = y^2 e^{xy}$$

$$\frac{\partial f}{\partial y} = ye^{xy} \cdot x + e^{xy} = xye^{xy} + e^{xy}.$$

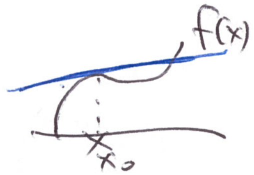
Example) $f(x,y) = \dfrac{\sin(xy)}{x^2+\cos(y)}$.

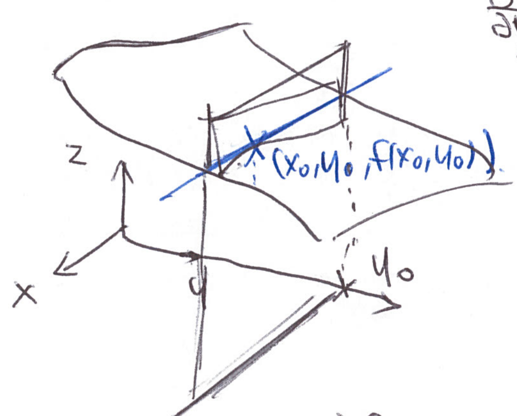
$$\frac{\partial f}{\partial x} = \frac{y\cos(xy)(x^2+\cos(y)) - 2x(\sin(xy))}{(x^2+\cos(y))^2}$$

※ Geometric Interpretation.

Recall!! Derivative of $y=f(x)$ at $x=x_0$

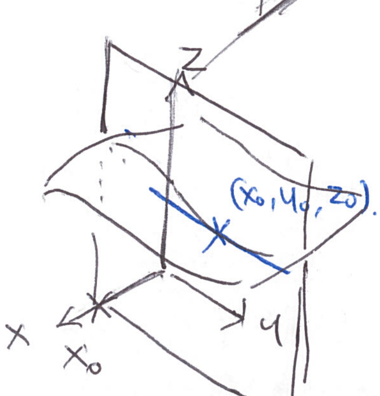
$\Rightarrow$ slope of the tangent line $f(x)$ at $x=x_0$





$\frac{\partial f}{\partial x}(x_0, y_0)$: partial derivative of $f(x,y)$ at (x_0, y_0)

$\Rightarrow$ the slope of the tangent line to the curve $\underbrace{z = f(x, y_0)}_{\text{function of } x}$. at $(x_0, y_0, f(x_0, y_0))$.



$\frac{\partial f}{\partial y}(x_0, y_0)$ = partial derivative of f at (x_0, y_0).

$\Rightarrow$ the slope of the tangent line to the curve $\underbrace{z = f(x_0, y)}_{\text{function of } y}$ at $(x_0, y_0, f(x_0, y_0))$.

* Higher-order Partial Derivatives.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

f_{xx} $\qquad$ f_{yy}.

Also, we have mixed derivatives.

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

Example) $f(x,y) = \sin x + xe^y$.

Find $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, f_{xy}, f_{yx}.

$f_{xx} = \frac{\partial}{\partial x}(\cos x + e^y) = -\sin x$

$f_{xy} = \frac{\partial}{\partial y}(\cos x + e^y) = e^y$.

$f_{yx} = \frac{\partial}{\partial x}(xe^y) = e^y$

the same

* Open disk.

$$B_r(x_0,y_0) = \{ (x,y) \text{ in } \mathbb{R}^2 \mid \sqrt{(x-x_0)^2 + (y-y_0)^2} < r \}.$$

: an open disk with radius r centered at (x_0, y_0).

* The mixed-derivative Theorem.

f : continuous.

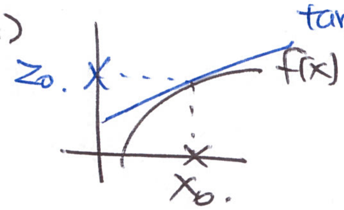
f_x, f_y, f_{xy}, f_{yx} : continuous on some open disk at (x_0,y_0).

$$\Rightarrow \quad f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$$

§10.4. Tangent planes, Differentiability, and Linearization.

Recall,

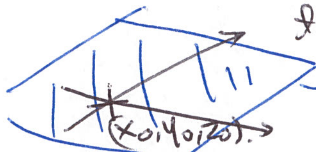
1)



tangent line.

$z - z_0 = f'(x_0)(x - x_0)$.

2). how to make a plane?



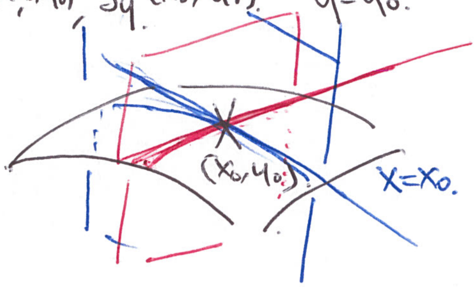
two lines. $z = mx + n$, $z = ly + k$ intersects at (x_0, y_0, z_0)

spanning one plane.

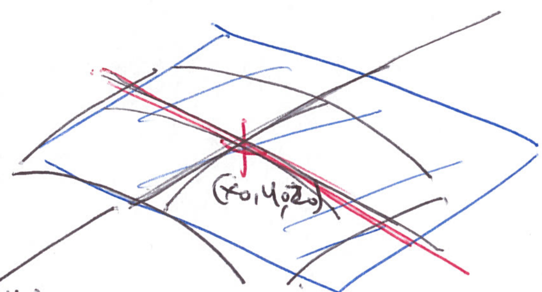
$z - z_0 = A(x - x_0) + B(y - y_0)$

$A = m$, $B = l$.

3) $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0)$. $y = y_0$.



* tangent planes.



Spanned by two tangent lines.

$z - z_0 = \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$. $z = f(x, y)$.

$z_0 = f(x_0, y_0)$.

$$z - z_0 = \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0).$$

Example). Find the equation of the tangent plane of

$$z = f(x,y) = 4x^2 + y^2$$

at $(1,2,8)$.

$$\frac{\partial f}{\partial x} = 8x, \quad \frac{\partial f}{\partial y} = 2y.$$

$$\Rightarrow \frac{\partial f}{\partial x}(1,2) = 8, \quad \frac{\partial f}{\partial y}(1,2) = 4$$

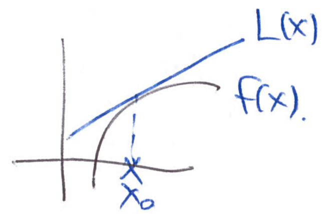
$$\Rightarrow \quad z - 8 = 8(x-1) + 4(y-2).$$

※ Differentiability.

Recall,, $f(x)$: differentiable at $x = x_0$.

$L(x) = f(x_0) + f'(x_0)(x - x_0)$: tangent line at $x = x_0$

linear approximation at $x = x_0$



$$\frac{|f(x) - L(x)|}{|x - x_0|} = \left| \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right|.$$

if $\displaystyle \lim_{x \to x_0} \left| \frac{f(x) - L(x)}{x - x_0} \right| = 0$,

then we say f is differentiable at x_0.

$f(x,y)$: $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are defined.

$$L(x,y) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0).$$

If $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \left| \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right| = 0$,

we say $f(x,y)$ is differentiable at (x_0, y_0).

$f(x,y)$: differentiable $\Rightarrow$ f: cont. ($\Leftrightarrow$ f: not cont $\Rightarrow$ f not diff'ble)

Example). $f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0. \end{cases}$

Show that $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$ exist.

but $f(x,y)$ is not continuous

and so not differentiable at $(0,0)$.

$\frac{\partial f(0,0)}{\partial x} = 0$. Since $f(x,0) = 0$. for all x.

$f(0,y) = 0$ for all $y \Rightarrow \frac{\partial f(0,0)}{\partial y} = 0$.

Continuity:

Let C_1 : the path $y=0$.

$\Rightarrow xy = 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} f(x,y) = 0.$$

Let C_2 : the path $x=y$.

$\Rightarrow xy = 1 \neq 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} f(x,y) = 1.$$ not cont.

* Sufficient condition for differentiability.

$f(x,y)$: defined on an open disk at (x_0,y_0).

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$: cont on an open disk at (x_0,y_0).

$\Rightarrow$ $f(x,y)$: differentiable at (x_0,y_0).

Example). Show that $f(x,y) = 2x^2y - y^2$ is differentiable for all (x,y) in $\mathbb{R}^2$.

$f(x,y)$ is defined on $\mathbb{R}^2$.

$\frac{\partial f}{\partial x} = 4xy$: continuous on $\mathbb{R}^2$.

$\frac{\partial f}{\partial y} = 2x^2 - 2y$: continuous on $\mathbb{R}^2$.

* Linearization.

$f(x,y)$: differentiable at (x_0,y_0).

Then,
$$L(x,y) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)$$

is called the linearization at (x_0,y_0).

If we approximate $f(x_0,y_0) \approx L(x_0,y_0)$,

it is called the standard linear approximation.

Example). Find the linearization of

$$f(x,y) = x^2y + 2xe^y$$

at $(2,0)$.

$$L(x,y) = 4 + 4(x-2) + 8y.$$

$$\frac{\partial f}{\partial x} = 2xy + 2e^y$$

$$\Rightarrow 2.$$

$$\frac{\partial f}{\partial y} = x^2 + 2xe^y$$

$$\Rightarrow 8.$$

Example). Approximate $f(3.05,\ 0.95)$.

for $f(x,y) = \ln(x - 2y^2)$.

$$\frac{\partial f}{\partial x} = \frac{1}{x-2y^2} \qquad \frac{\partial f}{\partial y} = \frac{-4y}{x-2y^2}$$

$$\frac{\partial f}{\partial x}(3,1) = 1, \quad \frac{\partial f}{\partial y}(3,1) = -4.$$

$$f(3,1) = 0.$$

$$L(x,y) = 0 + (x-3) - 4(y-1).$$

$$L(3.05, 0.95) = \underline{0.05 - 4(-0.05)}. = 0.25.$$

* Vector-valued Functions.

ex) $f(x,y) = \ln(x-2y^2)$

$f: \mathbb{R}^2 \to \mathbb{R}$. (Real-valued functions)

→ extend this to

$f: \mathbb{R}^n \to \mathbb{R}^m$.

vector-valued functions

$$f(x_1, x_2, \cdots, x_n) = \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ \vdots \\ f_m(x_1, \cdots, x_n) \end{bmatrix}$$

$f: \mathbb{R}^n \to \mathbb{R}^m$.

ex). $f(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$

where $\begin{cases} u(x,y) = x^2y - y^3 \\ v(x,y) = 2x^3y^2 + y. \end{cases}$

* Jacobi matrix. (for $f: \mathbb{R}^2 \to \mathbb{R}^2$).

$f: \mathbb{R}^2 \to \mathbb{R}^2$

with $f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$.

Then $(Df)(x_0, y_0) = \begin{bmatrix} \dfrac{\partial f_1(x_0,y_0)}{\partial x} & \dfrac{\partial f_1(x_0,y_0)}{\partial y} \\ \dfrac{\partial f_2(x_0,y_0)}{\partial x} & \dfrac{\partial f_2(x_0,y_0)}{\partial y} \end{bmatrix}$

is called <u>the Jacobi matrix</u>

Example) Find Df for $f(x,y) = \begin{bmatrix} x^2y - y^3 \\ 2x^3y^2 + y \end{bmatrix}$ at $(1,2)$

$$Df = \begin{bmatrix} 2xy & x^2-3y^2 \\ 6x^2y^2 & 4x^3y+1 \end{bmatrix}$$

$$Df(1,2) = \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix}.$$

* Linearization for vector-valued functions.

<u>Recall</u>: $f(x,y)$: real-valued functions

$$L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

vector-valued function.

$$f(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}.$$

$$L(x,y) = \begin{bmatrix} u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) \\ v(x_0,y_0) + \frac{\partial v}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial v}{\partial y}(x_0,y_0)(y-y_0) \end{bmatrix}.$$

$$= \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) \\ \frac{\partial v}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial v}{\partial y}(x_0,y_0)(y-y_0) \end{bmatrix}$$

$$= \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x}(x_0,y_0) & \frac{\partial u}{\partial y}(x_0,y_0) \\ \frac{\partial v}{\partial x}(x_0,y_0) & \frac{\partial v}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$$= \begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + Df(x_0,y_0) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}.$$

Example)

$$f:(x,y)=\begin{bmatrix} ye^{-x} \\ \sin x+\cos y \end{bmatrix}.$$

Approximate $f(0.1, -0.1)$.

$$Df(x,y)=\begin{bmatrix} -ye^{-x} & e^{-x} \\ \cos y & -\sin y \end{bmatrix} \qquad f(0,0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Df(0,0)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$L(x,y)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 \\ 1 \end{bmatrix}+\begin{bmatrix} y \\ x \end{bmatrix}=\begin{bmatrix} y \\ x+1 \end{bmatrix}.$$

$$L(0.1,-0.1)=\begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix}.$$

(actually, $f(0.1,-0.1)=\begin{bmatrix} -0.09 \\ 1.09 \end{bmatrix}$).