Ch.7 Transcendental Functions

§7.1 Review of Log & Exp

* The natural logarithm.

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad , \quad x>0$$

by FTC, $\frac{d}{dx}\ln x = \frac{1}{x}$.

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x).$$

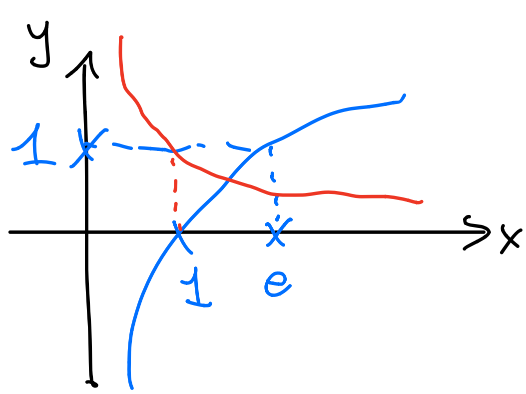
ex) $\frac{d}{dx}\ln(3x) = \frac{1}{3x}\cdot 3 = \frac{1}{x}$.

* e.

: The number satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

$(e \approx 2.71828\cdots)$

* 

domain $(x>0)$

range $(-\infty < x < \infty)$.

$*\ \int \frac{1}{u}\,du.$

ex) $\frac{d}{dx}\ln x = \frac{1}{x}$, $\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{d}{dx}\ln x & \text{if } x \geq 0 \\ \frac{d}{dx}\ln(-x) & \text{if } x < 0. \end{cases}$$

$$= \frac{1}{x}$$

$\Rightarrow \int \frac{1}{u}\,du = \ln|u| + C$

if $u = f(x)$ $\Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$

$du = f'(x)\,dx$

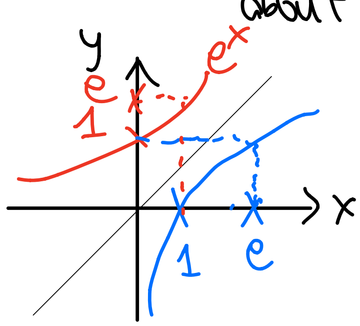
$*$ Inverse of $\ln x$.

Recall, If $(f \circ g)(x) = x$ (identity),

then $g\,(= f^{-1})$ is called an inverse of f

The graph of $f^{-1}(x)$ is symmetric

about the line $y = x$.



domain $(-\infty, \infty)$

range $(0, \infty)$

exponential function with the base e.

$e^x = \exp(x)$: the natural exponential function.

* Derivative & Integral of e^x.

<u>Recall</u>, $\frac{d}{dx}a^x = a^x \ln a$

$$\frac{d}{dx}e^x = e^x \quad \Rightarrow \int e^x dx = e^x + C$$

§7.2 Separable Differential Equations.

Recall: A differential equation is an equation that contains one or more derivatives.

ex) $\frac{dy}{dx} = 2$

Solving : find the function $y = f(x)$ satisfying a DE.

ex) $y = 2x + 1$

order: the highest order of the derivative in a DE.

* separable.

: a DE with a form $\frac{dy}{dx} = p(x)\,q(y)$.

ex) $\frac{dy}{dx} = xy$ $\quad p(x) = x, \quad q(y) = y.$

$\frac{dy}{dx} = 3x$ $\quad p(x) = 3x, \quad q(y) = 1$

$\frac{dy}{dx} = 3x^3y + 4\sqrt{x}$ $\quad$ No.

$\frac{dy}{dx} = e^{3x+y} = e^{3x} \cdot e^{y}$ $\quad p(x) = e^{3x}, \quad q(y) = e^{y}$

$\frac{dy}{dx} = xy + 3x - 2y - 6$

$= y(x-2) + 3(x-2)$

$= (x-2)(y+3)$ $\quad p(x) = x-2, \quad q(y) = y+3$

* solving a separable DE.

① Find $p(x)$ & $q(y)$.

ex) $\frac{dy}{dx} = xy$ $\quad p(x) = x \;,\; q(y) = y.$

② Separate the variable.

"make $\frac{1}{q(y)} dy = p(x) dx$."

ex) $\frac{1}{y} dy = x dx$.

③ Integrate the both sides.

ex) $\int \frac{1}{y} dy = \int x \, dx$

$\Rightarrow \ln(y) + C = \frac{1}{2}x^2$.

④ solve for y.

ex) $\ln|y| = \frac{1}{2}x^2 - C \Rightarrow e^{\ln|y|} = |y|^{\ln e} = |y| = e^{\frac{1}{2}x^2 - C} = e^{\frac{1}{2}x^2} \cdot e^{-C}$

$= e^{\frac{1}{2}x^2} \cdot C$

$\Rightarrow y = \pm e^{\frac{1}{2}x^2} \cdot C$.

* particular solution.

: a specific function satisfying DE & the initial condition.

ex) $\frac{dy}{dx} = xy$, $y(0) = 2$, $y > 0$.

$$y = e^{\frac{1}{2}x^2} \cdot c \underset{\substack{x=0 \\ y=2}}{\Longrightarrow} 2 = c$$

$$\Rightarrow y = 2e^{\frac{1}{2}x^2} \quad \text{(particular solution)}$$

ex) $\frac{dy}{dx} = y^2 e^{3x}$, $y(0) = 1$

$$\frac{1}{y^2} dy = e^{3x} dx \Rightarrow \int y^{-2} dy = \int e^{3x} dx$$

$$\Rightarrow -y^{-1} = \frac{1}{3}e^{3x} + C$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{3}e^{3x} + C \Rightarrow \frac{1}{y} = -\frac{1}{3}e^{3x} + C$$

$$\Rightarrow y = \frac{1}{-\frac{1}{3}e^{3x} + C} \underset{\substack{x=0 \\ y=1}}{\Longrightarrow} 1 = \frac{1}{-\frac{1}{3} + C}$$

$$\Rightarrow C = \frac{4}{3} \Rightarrow y = \frac{1}{-\frac{1}{3}e^{3x} + \frac{4}{3}} = \frac{1}{\frac{-e^{3x}+4}{3}}$$

$$= \frac{3}{-e^{3x}+4}$$

ex) $\frac{dy}{dx} = xy + 3x - 2y - 6$

$= (x-2)(y+3)$

$\Rightarrow \int \frac{1}{y+3}\, dy = \int (x-2)\, dx$

$\Rightarrow \ln|y+3| = \frac{1}{2}x^2 - 2x + C.$

$\Rightarrow |y+3| = e^{\frac{1}{2}x^2 - 2x} \cdot C$

$\Rightarrow y+3 = \pm e^{\frac{1}{2}x^2 - 2x} \cdot C \Rightarrow y = \pm e^{\frac{1}{2}x^2 - 2x} \cdot C - 3$

Ch. 8 Techniques of Integration.

§8.2. Integration by parts.

* Integration by parts.

Recall "product Rule"

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

$$\Rightarrow \int f(x)g'(x) = \int [f(x)g(x)]' - \int f'(x)g(x)$$

$$= \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

1. Find the f & $g'dx$.
2. Integrate $g'dx \Rightarrow g$.
3. differentiate $f \Rightarrow f'dx$

* Order in which to choose $f(x)$.

I : Inverse function

L : Log

A : Algebraic (Poly, rational, ...)

T : Trigonometric

E : exp.

ex) $\int \sin^{-1} x \, dx$

$f(x) = \sin^{-1} x, \quad g'(x) = 1.$

$\Rightarrow \int \sin^{-1} x \, dx = (\sin^{-1} x) \cdot x - \int \frac{1}{\sqrt{1-x^2}} x \, dx$

$= x \sin^{-1} x + \int \frac{1}{2} \cdot \frac{1}{\sqrt{t}} dt$ $\quad$ $1-x^2 = t$, $-2x\,dx = dt$

$= x \sin^{-1} x + \left(\frac{1}{2} \cdot 2 t^{\frac{1}{2}} + C\right)$

$= x \sin^{-1} x + \sqrt{1-x^2} + C.$

ex) $\int x^2 e^{-x} \, dx$

$f(x) = x^2, \quad g'(x) = e^{-x}.$

$\Rightarrow x^2 \cdot (-e^{-x}) + \int 2x e^{-x} dx$ $\quad$ $f(x) = x$, $g'(x) = e^{-x}$

$= -x^2 e^{-x} + 2\left[x(-e^{-x}) + \int e^{-x} dx\right]$

$= -x^2 e^{-x} - 2x e^{-x} - e^{-x} + C$

ex) $\int \sin[\ln x]\, dx$

$f(x) = \sin[\ln x] \qquad g'(x) = 1.$

$\Rightarrow \ x\sin[\ln x] - \int x \cdot \cos[\ln x] \cdot \frac{1}{x}\, dx$

$= x\sin[\ln x] - \int \cos[\ln x]\, dx$

$f(x) = \cos[\ln x]$

$g'(x) = 1.$

$= x\sin[\ln x] - \left[x\cos[\ln x] - \int x\left[-\sin[\ln x] \cdot \frac{1}{x}\right] dx \right]$

$= x\sin[\ln x] - x\cos[\ln x] - \int \sin[\ln x]\, dx.$

$\Rightarrow \int \sin[\ln x]\,dx = x\sin[\ln x] - x\cos[\ln x]$
$- \int \sin[\ln x]\, dx$

$\Rightarrow \ 2\int \sin[\ln x]\, dx = x\sin[\ln x] - x\cos[\ln x] + C$

$\Rightarrow \int \sin[\ln x]\, dx = \frac{1}{2}x\sin[\ln x] - \frac{x}{2}\cos[\ln x] + C.$

$\boxed{\int e^x \sin x\, dx}$

§ 8.3 Powers & Products of Trigonometric Functions

<u>Recall</u> Trig identities.

(a) $\cos^2 x + \sin^2 x = 1$

(b) $1 + \tan^2 x = \sec^2 x$

(c) $\sin^2 x = \frac{1}{2}[1 - \cos(2x)]$

(d) $\cos^2 x = \frac{1}{2}[1 + \cos(2x)]$

(e) $\sin(2x) = 2\sin x \cos x$

ex) $\int \tan^3 x \, dx$ –

$\int \tan x (\sec^2 x - 1) \, dx = \int [\tan x \sec^2 x - \tan x] \, dx$

$\tan x = t$.

$\sec^2 x \, dx = dt$.

$= \int t \, dt - \int \tan x \, dx = \frac{1}{2} t^2 - \int \frac{\sin x}{\cos x} \, dx$

$\cos x = s \qquad -\sin x \, dx = ds$

$= \frac{1}{2} t^2 + \int \frac{ds}{s} = \frac{1}{2} t^2 + \ln|s| + C$

$= \frac{1}{2} \tan^2 x + \ln|\cos x| + C$

ex) $\int \sin^4 x \, dx$

$$= \int \left\{ \frac{1}{2} [1 - \cos 2x] \right\}^2 dx$$

$$= \int \frac{1}{4} [1 - 2\cos 2x + \cos^2 2x] \, dx$$

$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x \, dx + \frac{1}{4} \int \cos^2 2x \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{4} \int \frac{1}{2} [1 + \cos 4x] \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$$

ex) $\int \tan^3 x \sec^3 x \, dx$

$$\sec x = t$$

$$\sec x \tan x \, dx = dt$$

$$\int (\sec^2 x - 1) \tan x \sec x \cdot \sec^2 x \, dx$$

$$= \int (t^2 - 1) t^2 \, dt = \int (t^4 - t^2) \, dt$$

$$= \frac{1}{5}t^5 - \frac{1}{3}t^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$