Rank 2 symmetric matrices, tropicalization & algebraic matroid

joint work with May Cai & Josephine Yu

Kisun Lee
Clemson University

Tropical algebra

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tropical semiring

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tropical matrix

: a matrix with entries in the tropical semiring

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(tropical rank)

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: a (Laurent) polynomial

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(tropical hypersurface)

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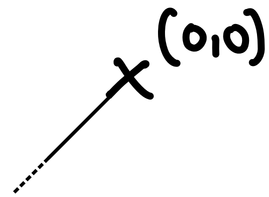

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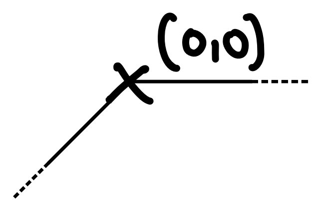

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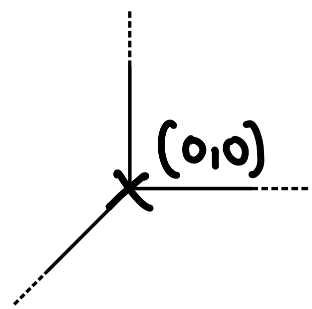

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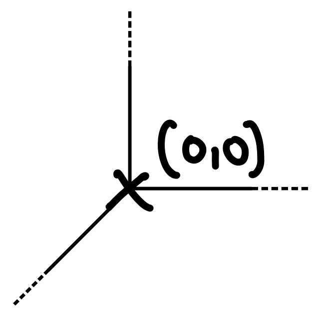

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$$\mathrm{trop}(\mathbb{V}) = \bigcap_{f \in I} \mathrm{trop}(V(f)) \quad \text{(tropical variety)}$$

Tropical varieties

(Maclagan - Sturmfels textbook Theorem 3.3.5)

if V : irreducible of d-dimensional,

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(tropical variety has a polyhedral structure)

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3x3 minor

$-x_{13}x_{22}x_{31} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{11}x_{22}x_{33}$

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$\Rightarrow$ tropical rank 2

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symmetric tropical rank 3

(even though it is tropical rank 2)

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symmetric tropical rank 2

Q. How to represent tropical rank 2 matrices combinatorially?

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A. Tropical convexity

Tropical Convexity

$S \subset \mathbb{R}^n$ is called **tropically convex**

if for any $x, y \in S$, $a, b \in \mathbb{R}$

$$a \odot x \oplus b \odot y \in S$$

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<u>Remark</u>. If S is tropically convex,
then $S + \mathbb{R}\mathbb{1} \subset S$
Hence we work on $\mathbb{R}^n / \mathbb{R}\mathbb{1}$

(Develin, Santos, Sturmfels 2005)

M: an $n \times d$ tropical matrix

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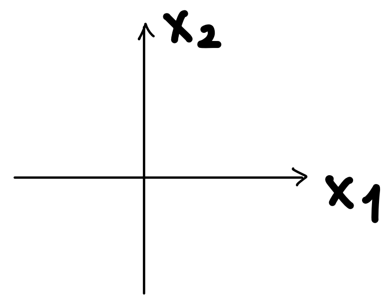

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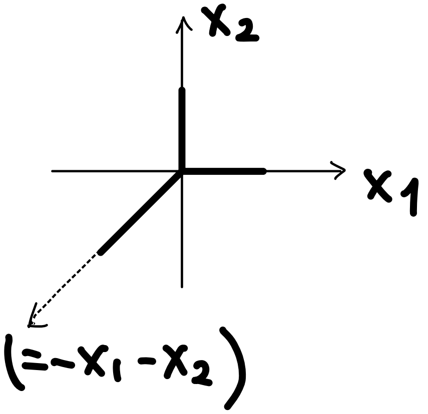

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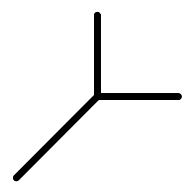

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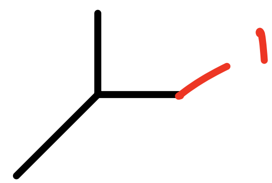

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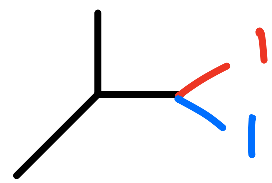

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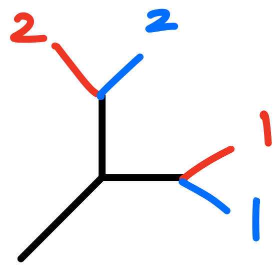

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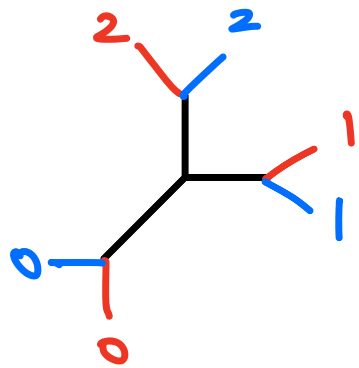

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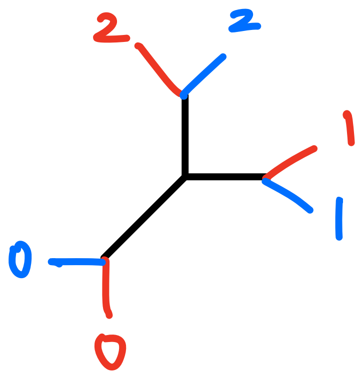

(Cai, L., Yu)

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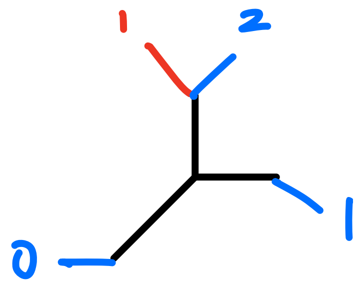

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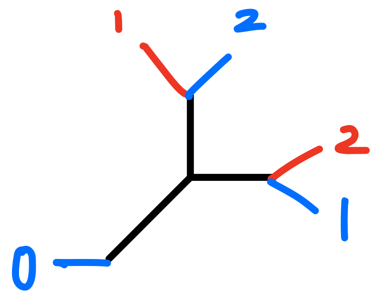

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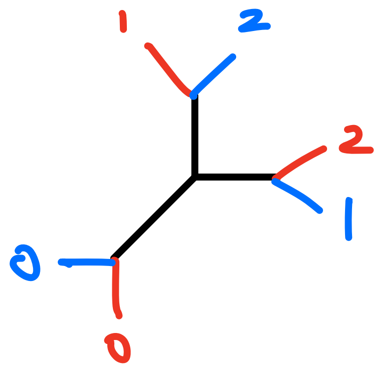

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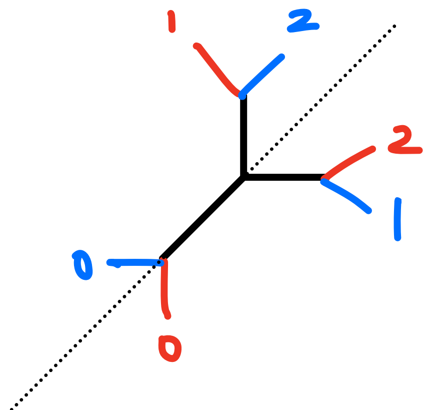

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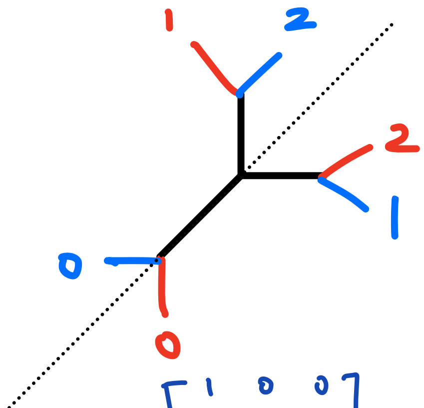
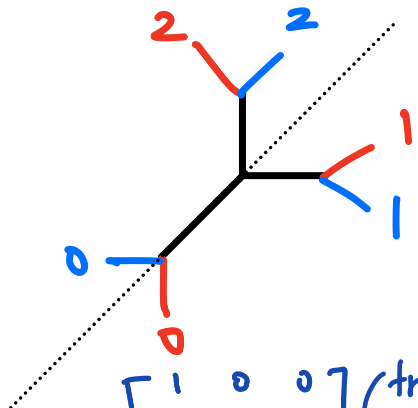
ex)



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (tropical rank 2, symmetric tropical rank 3)

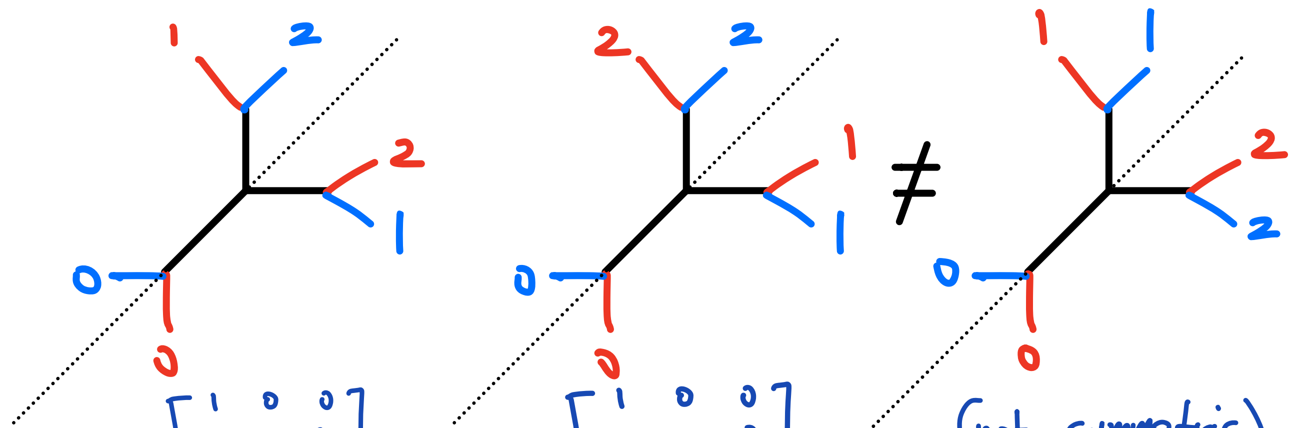
(Cai, L., Yu)

the space of symmetric trop rank 2

form a simplicial fan structure of

symmetric bicolored trees
(symbic trees)

ex)



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

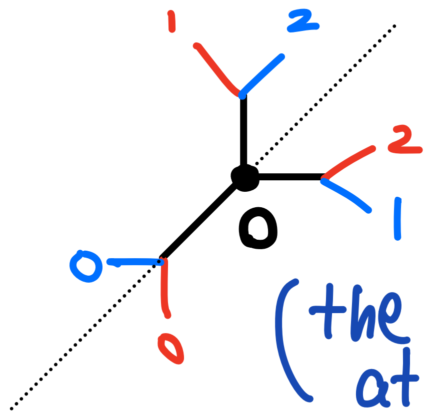
(not symmetric)

(Cai, L., Yu)

the space of symmetric trop rank 2

form a simplicial fan structure of symmetric bicolored trees (symbic trees)

ex) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$



(the point 0 located at the origin)

(Cai, L., Yu)

the space of symmetric trop rank 2

form a simplicial fan structure of

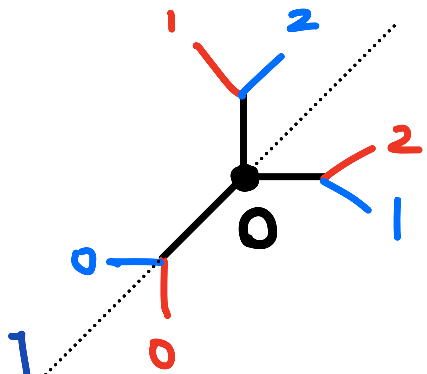
symmetric bicolored trees
(symbic trees)

ex)

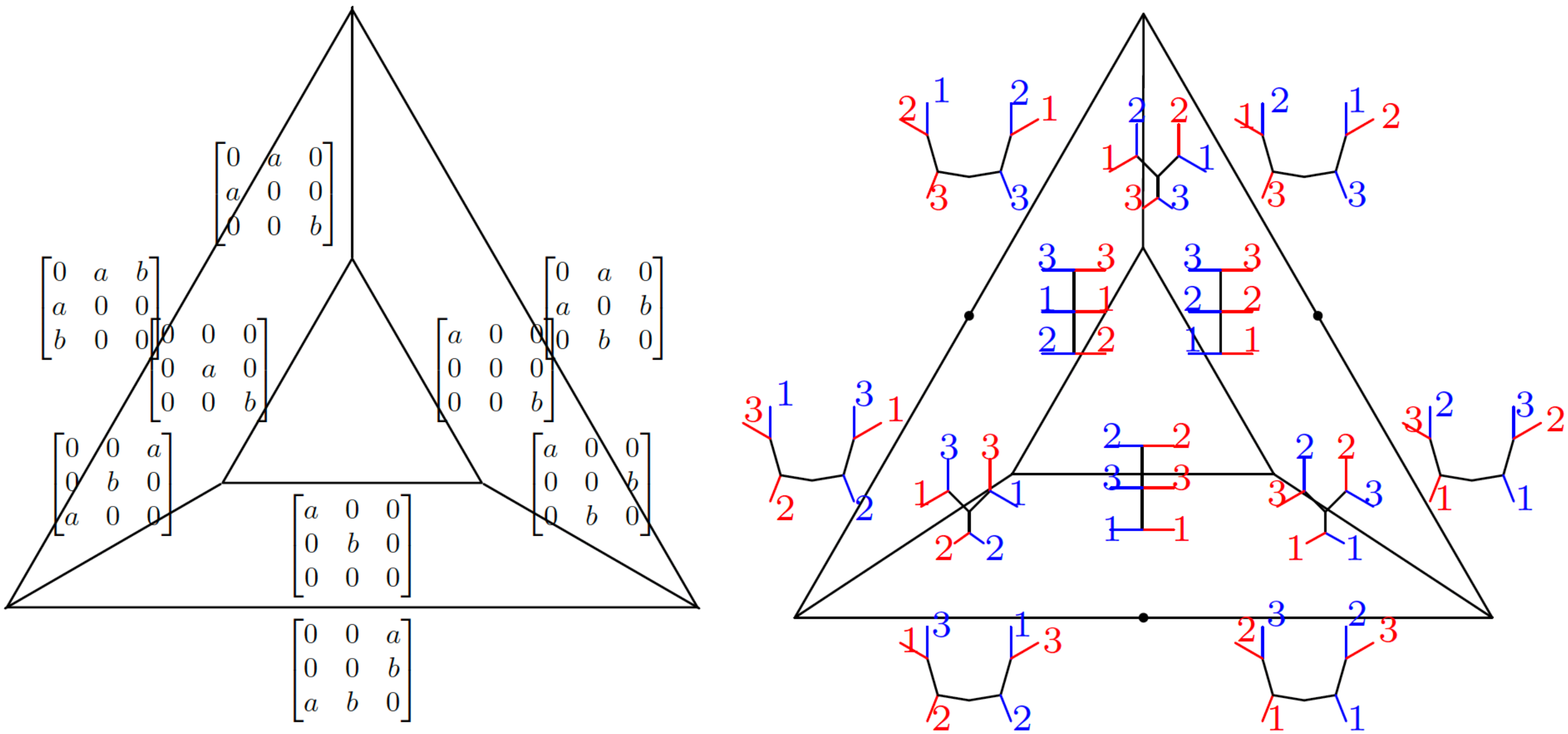
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} 2d & d+e & d+f \\ d+e & 2e & e+f \\ d+f & e+f & 2f \end{bmatrix}$$



(translating the point O)



The space of 3x3 symmetric tropical rank 2 matrices

Shellability

Shellability

Shellability

Can we peel the simplicial complex of symmetric tropical rank 2 matrices without breaking it?

Shellability

(shelling)

A shelling of a pure-dimensional simplicial complex is a total ordering $<$ on the facets so that $\forall$ two facets $C' < C$ there exists another facet C'' such that

1) $C' \cap C \subseteq C'' \cap C$
2) $C'' < C$
3) $C \setminus C''$ is a vertex of C

Shellability

(Markwig, Yu 2009)

the space of rank 2 matrices is shellable

Shellability

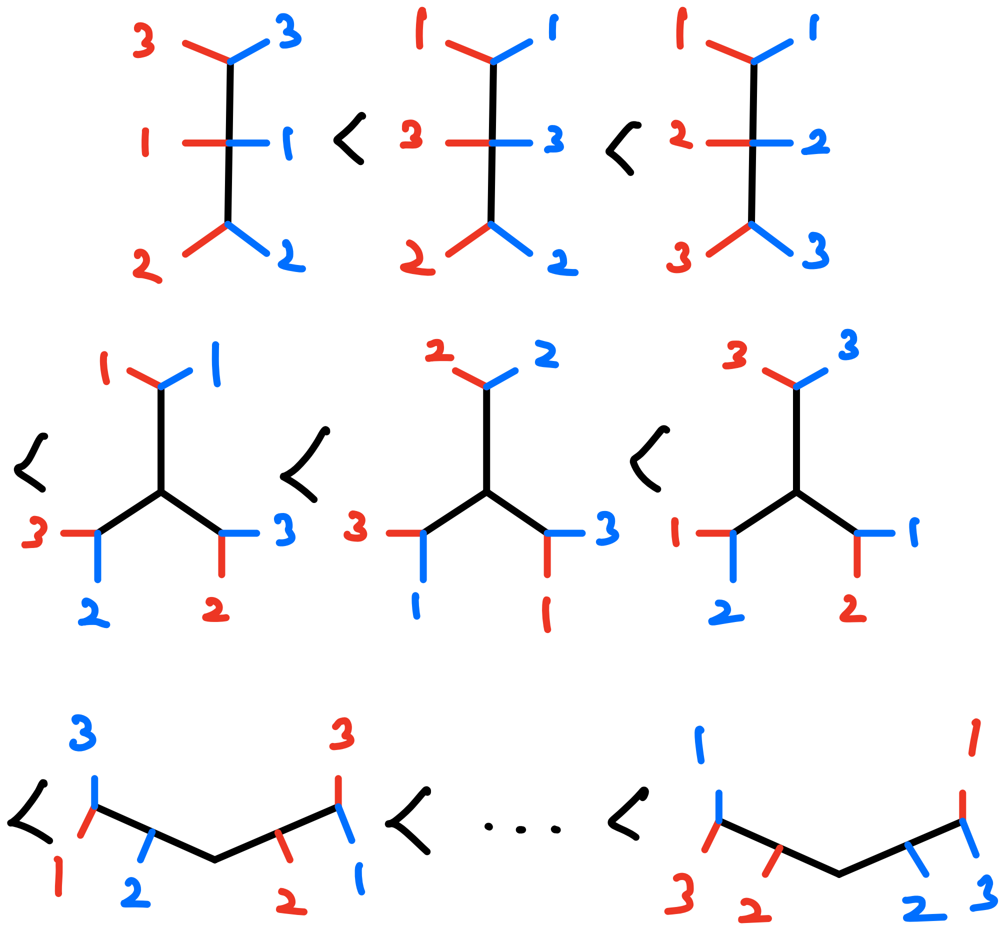
(Markwig, Yu 2009)

the space of rank 2 matrices is shellable

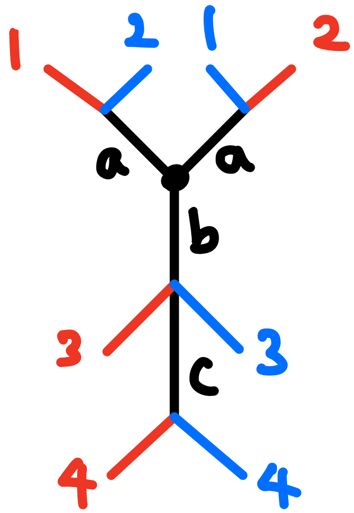
(Cai, L., Yu)

the space of symmetric trop rank 2 is shellable

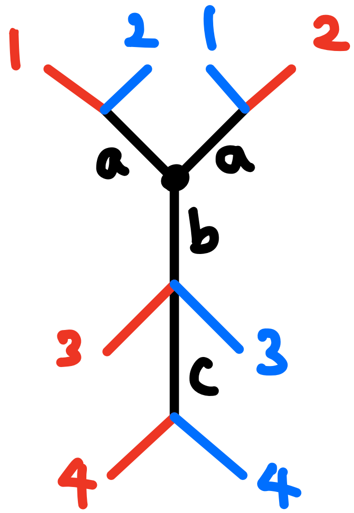
Shellability



The matroid of symbic trees



The matroid of symbic trees

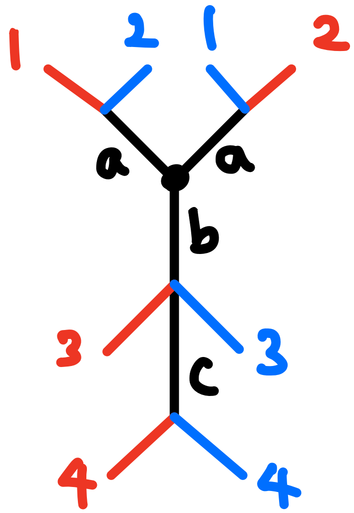


(symmetric tropical rank 2 matrix)

$$\begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & b & b+c \end{bmatrix} + \begin{bmatrix} 2d & d+e & d+f & d+g \\ d+e & 2e & e+f & e+g \\ d+f & e+f & 2f & f+g \\ d+g & e+g & f+g & 2g \end{bmatrix}$$

(parametrizing edge length) (translating the tree)

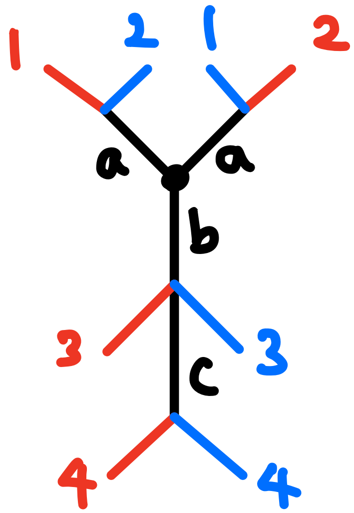
The matroid of symbic trees



$$\begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & b & b+c \end{bmatrix} + \begin{bmatrix} 2d & d+e & d+f & d+g \\ d+e & 2e & e+f & e+g \\ d+f & e+f & 2f & f+g \\ d+g & e+g & f+g & 2g \end{bmatrix}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{cccccccccc} 11 & 12 & 13 & 14 & 22 & 23 & 24 & 33 & 34 & 44 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

The matroid of symbic trees

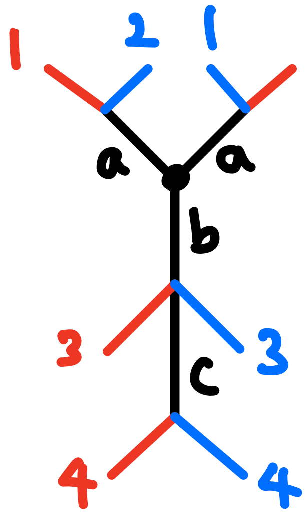


$$\begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & b & b+c \end{bmatrix} + \begin{bmatrix} 2d & d+e & d+f & d+g \\ d+e & 2e & e+f & e+g \\ d+f & e+f & 2f & f+g \\ d+g & e+g & f+g & 2g \end{bmatrix}$$

$$\begin{array}{c c} & \begin{array}{cccccccccc} 11 & 12 & 13 & 14 & 22 & 23 & 24 & 33 & 34 & 44 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

distance parameter matrix

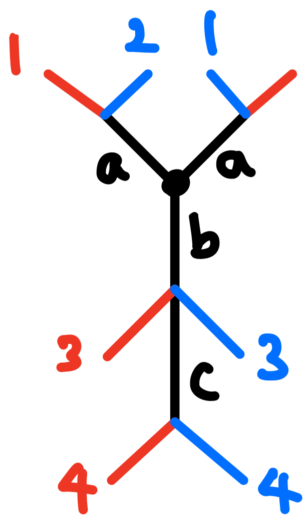
The matroid of symbic trees



$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2
\end{bmatrix}
$$

the linear matroid of the distance parameter matrix defines the matroid of a symbic tree

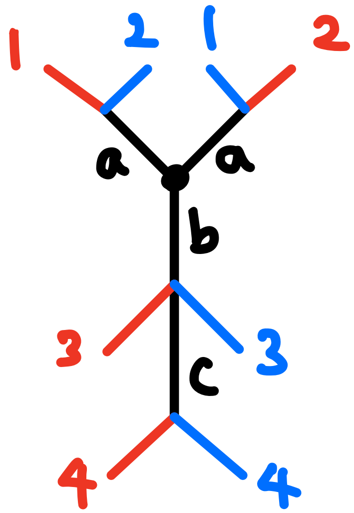
The matroid of symbic trees



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

bases of the matroid of symbic trees
characterize bases of
(regular) rank-2 symmetric matrices

The matroid of symbic trees



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

bases of the matroid of symbic trees
characterize bases of
(regular) rank-2 symmetric matrices

(Bernstein 2017) bases for rank-2 matrices

The matroid of symbic trees

(Cai, L., Yu)

The collection of bases in the algebraic matroid of rank-2 symmetric matrices is the union of bases of matroids of union of trees with caterpillar branches

The matroid of symbic trees

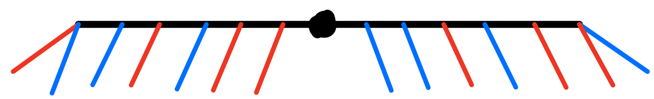
(Cai, L., Yu)

The collection of bases in the algebraic matroid of rank-2 symmetric matrices is the union of bases of matroids of union of trees with caterpillar branches

The matroid of symbic trees

(Cai, L., Yu)

The collection of bases in the algebraic matroid of rank-2 symmetric matrices is the union of bases of matroids of union of trees with caterpillar branches

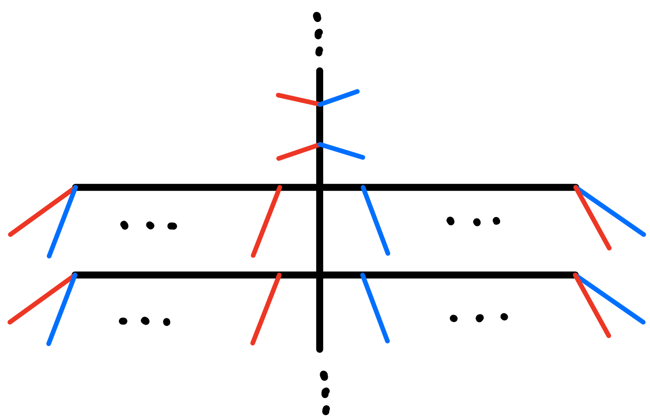


Caterpillar symbic tree

The matroid of symbic trees

(Cai, L., Yu)

The collection of bases in the algebraic matroid of rank-2 symmetric matrices is the union of bases of matroids of union of trees with caterpillar branches



Thank you

for your attention